We want to study under what conditions, weak convergence implies strong convergence.

Let $1 \le p < \infty$.
$f_k \in L^p(X, \mathcal{M}, \mu)$, $f \in L^p(X, \mathcal{M}, \mu)$.

$$f_k \rightharpoonup f \text{ (weakly)} \iff \int_X f_k g \, d\mu \to \int_X fg \, d\mu \quad \forall g \in L^{p'}(X, \mathcal{M}, \mu).$$

Thm: Let $(X, \mathcal{M}, \mu)$ a measure space. $f, f_k \in L^p(X, \mathcal{M}, \mu)$, $1 \le p < \infty$. Then
$$\|f_k - f\|_p \to 0 \implies f_k \rightharpoonup f \text{ weakly in } L^p.$$

Proof:
Let $g \in L^{p'}$
$$\implies \left| \int_X f_k g \, d\mu - \int_X fg \, d\mu \right| = \left| \int_X (f_k - f) g \, d\mu \right|$$
$$\le \int_X |f_k - f| \, |g| \, d\mu$$
$$\le \|f_k - f\|_p \|g\|_{p'} \to 0.$$

**Thm**: Let $1<p<\infty$.

$f_k \to f$ $\mu$-a.e., $f_k, f \in L^p(X)$

Then

$f_k \rightharpoonup f \iff \{\|f_k\|_p\}$ bounded sequence.

$\Longrightarrow$

Let $f_k \rightharpoonup f$. Thm 290.1 implies

$\|f_k\|_p \le M$, $k=1,2,3,\dots$

$\Longleftarrow$ Let $g \in L^{p'}(X,\mu)$.

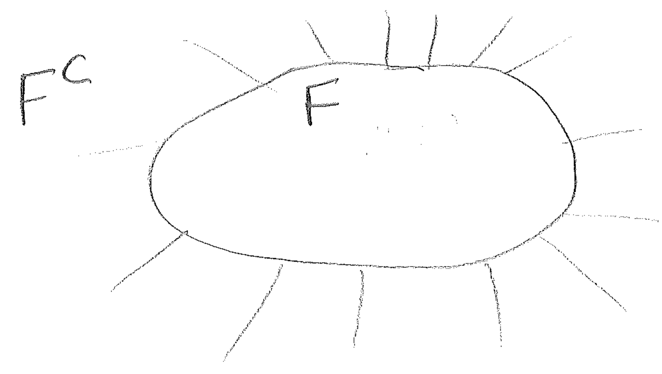
Let $\varepsilon>0$.

(A) $\forall \varepsilon>0$ $\exists \delta$ s.t $\mu(E)<\delta \Rightarrow \left(\int_E |g|^{p'} d\mu\right)^{1/p'} < \varepsilon$

Claim: $\exists F \in \mathcal{M}$ s.t $\mu(F)<\infty$ and

$\left(\int_{F^c} |g|^{p'} d\mu\right)^{1/p'} < \varepsilon$

Let $A_t := \{x : |g(x)|^{p'} \ge t\}$




$\|g\|_{p', F^c}$ very small.

Monotone convergence Theorem implies:

$$\lim_{t\to 0} \int_X \chi_{A_t} |g|^{p'} d\mu = \int_X |g|^{p'} d\mu$$

$$\Rightarrow \left| \int_X |g|^{p'} d\mu - \int_{A_t} |g|^{p'} d\mu \right| < \varepsilon , \quad t \le t_0$$

$$\Rightarrow \left| \int_{A_t^c} |g|^{p'} d\mu \right| < \varepsilon, \quad t \le t_0 .$$

Set $F := A_t , \quad t \le t_0 .$ ∎

$$\left| \int_X fg\, d\mu - \int_X f_k g\, d\mu \right| \le \int_X |f - f_k|\, |g|\, d\mu$$

$$= \int_F |f-f_k|\, |g|\, d\mu + \int_{F^c} |f-f_k|\, |g|\, d\mu$$

$$\le \int_F |f - f_k| |g| d\mu + \|f-f_k\|_{p, F^c} \|g\|_{p', F^c}$$

$$\le \int_F |f-f_k| |g|\, d\mu + \underbrace{(\|f\|_{p,F^c} + \|f_k\|_{p,F^c}) \|g\|_{p',F^c}}_{<\varepsilon}$$

Note: $\|f\|_p^p = \int_X |f|^p d\mu = \int_X \lim_{k\to\infty} |f_k|^p d\mu \le \lim_{k\to\infty} \int_X |f_k|^p d\mu \le M^p$.

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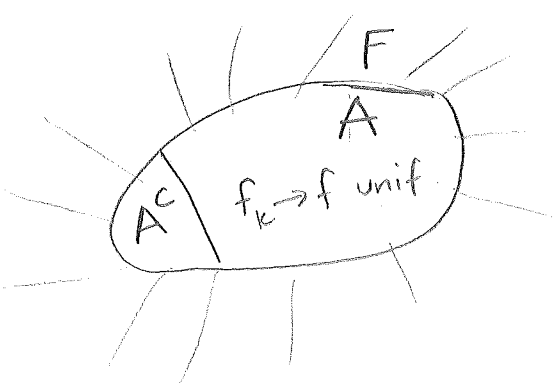
$\mu(A^c) < \delta$




$f_k \to f$ unif. on $A$ by Egoroff's Thm. This theorem requires $\mu(F) < \infty$.

$$\leq \int_F |f - f_k||g| \, d\mu + \varepsilon \cdot 2M$$

$$= \int_A |f - f_k| \, |g| \, d\mu + \int_{A^c} |f - f_k| \, |g| \, d\mu + 2M\varepsilon$$

$$\leq \underbrace{\|f - f_k\|_{p,A} \, \|g\|_{p',A}}_{①} + 2M \underbrace{\|g\|_{p',A^c}}_{②} + 2M\varepsilon$$

① $< \varepsilon$ : $f_k \to f$ unif on $A$

$\Rightarrow |f_k - f|^p \to 0 \quad \Rightarrow |f_k - f|^p \leq \varepsilon \quad \forall x \in A, \ \forall k \geq N$

$|f_k - f|^p$ is dominated by and integrable function $g = \varepsilon$, since $\int_A g \, d\mu = \varepsilon \cdot \mu(A) < \infty$.

$\therefore$ Lebesgue Dominated Convergence Thm $\Rightarrow$

$$\lim_{k \to \infty} \int_A |f_k - f|^p = \int_A \lim_{k \to \infty} |f_k - f|^p \, d\mu = 0$$

$\therefore \|f_k - f\|_p \to 0$

② $< \varepsilon$ because $\mu(A^c) < \delta$ and (A).

Thm: $1 \le p < \infty$.

$f, f_K \in L^p(X, \mathcal{M}, \mu)$

$f_K \to f$ $\mu$-a.e., $\|f_K\|_p \to \|f\|_p$

$\{\|f_K\|_p\}_{K=1}^{\infty}$ bounded

Then:

$f_K \to f$ in the strong topology;

i.e.

$\|f_K - f\|_p \to 0$.

Proof:

Previous Thm gives only

$f_K \rightharpoonup f$ (in the weak topology).

It can be proven (using $f_K \to f$ $\mu$-a.e. and $\|f_K\|_p \le M$) that:

$$\lim_{K \to \infty} \left( \|f_K\|_p^p - \|f_K - f\|_p^p \right) = \|f\|_p^p$$

But $\|f_K\|_p^p \to \|f\|_p^p$ gives:

$\|f_K - f\|_p^p \to 0$; i.e.

$\|f_K - f\|_p \to 0$.

# Heat Equation.

(a) Assume $n=1$ and $u(x,t)=v\left(\frac{x}{\sqrt{t}}\right)$. Show

$$u_t = u_{xx}$$

$$\Longleftrightarrow$$

$$z\,v'(z) + 2v''(z) = 0$$

(b) Show that the general solution of (*) is:

$$v(z) = c \cdot \int_0^z e^{-s^2/4}\,ds + d$$

(c) Differentiate $v\left(\frac{x}{\sqrt{t}}\right)$ with respect to $x$ and select the constant $c$ properly, so as to obtain the fundamental solution $\Phi$ for $n=1$.

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$$u(x,t) = v\left(\frac{x}{\sqrt{t}}\right)$$

$$u_t = \frac{\partial}{\partial t}\left(v\left(x t^{-1/2}\right)\right)$$

$$= v'\left(\frac{x}{\sqrt{t}}\right) \cdot -\frac{1}{2} x t^{-3/2}$$

$$= -\frac{1}{2} v'\left(\frac{x}{\sqrt{t}}\right) \frac{x}{t^{3/2}}$$

$$u_x = \frac{\partial}{\partial x}\left(v\left(\frac{x}{\sqrt{t}}\right)\right) =$$

$$= v'\left(\frac{x}{\sqrt{t}}\right)\left(\frac{1}{\sqrt{t}}\right)$$

$$u_{xx} = v''\left(\frac{x}{\sqrt{t}}\right) \cdot \frac{1}{t}$$

$$u_t - u_{xx} = 0 \iff -\frac{1}{2} \frac{x}{t^{3/2}} v'\left(\frac{x}{\sqrt{t}}\right) - \frac{1}{t} v''\left(\frac{x}{\sqrt{t}}\right) = 0$$

$$\iff \frac{1}{t}\left[\left(\frac{x}{2 t^{1/2}}\right) v'\left(\frac{x}{\sqrt{t}}\right) + v''\left(\frac{x}{\sqrt{t}}\right)\right] = 0$$

$$\iff v'(z) \cdot \frac{z}{2} + v''(z) = 0$$

$$v'(z)\, z + 2 v''(z) = 0$$

Let $u(z) = v'(z)$

$u'(z) = v''(z)$

$$u(z)\, z + 2 u'(z) = 0$$

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$$2\frac{du}{dz} = -u(z)\,z$$

$$\int 2\,\frac{du}{u} = \int -z\,dz$$

$$2\ln|u(z)| = -\frac{z^2}{2} + K$$

$$\ln|u(z)| = -\frac{z^2}{4} + K$$

$$|u(z)| = e^{-z^2/4+K} = c\,e^{-z^2/4}$$

$$u(z) = c\,e^{-z^2/4}$$

$$v'(z) = u(z) = c\,e^{-z^2/4}$$

$$\int_0^z v'(s)\,ds = \int_0^z c\,e^{-s^2/4}\,ds + d$$

$$v(z) = \int_0^z c\,e^{-s^2/4}\,ds + d$$

(b) $$\boxed{u(t,x) = v\left(\frac{x}{\sqrt{t}}\right) = \int_0^{x/\sqrt{t}} c\,e^{-s^2/4}\,ds \quad \text{sol. to } u_t = u_{xx}.}$$

(c) $$\frac{\partial u}{\partial x} = c\,e^{-\frac{x^2}{4t}} \cdot \frac{1}{\sqrt{t}} = \frac{1}{(4\pi t)^{1/2}}\,e^{\frac{-x^2}{4t}}$$

$$\therefore \quad c = \frac{1}{\sqrt{4\pi}}$$

Consider

$$u_t = \Delta u \quad \text{in} \quad \mathbb{R}^n \times (0,\infty)$$
$$u = g \quad \text{on} \quad \mathbb{R}^n \times \{t=0\}$$

$(x,t) \mapsto \Phi(x,t)$ solves the heat equation away from $(0,0)$, and the same is true for $(x,t) \longmapsto \Phi(x-y,t)$ for each fixed $y$. The convolution

$$(*) \quad u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)\, g(y)\, dy$$
$$= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy, \quad x \in \mathbb{R}^n, \; t>0$$

is also a solution.

<u>Thm</u>: Let $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then $u$ given in $(*)$ satisfies.

(i) $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$.

(ii) $u_t(x,t) - \Delta u(x,t) = 0 \quad (x \in \mathbb{R}^n, t>0)$

(iii) $\displaystyle\lim_{(x,t)\to(x^0,0)} u(x,t) = g(x^0), \quad \forall x^0 \in \mathbb{R}^n.$

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Double check that

$$u_t = u_{xx}$$

with $u(t,x) = \int_0^{x/\sqrt{t}} c\, e^{-s^2/4}\, ds.$

$$u_t = c\, e^{-\frac{x^2}{4t}} x \left(-\tfrac{1}{2} t^{-3/2}\right) = -\frac{c}{2} x t^{-3/2} e^{-x^2/4t}$$

$$u_x = c\, e^{-\frac{x^2}{4t}} \cdot \frac{1}{\sqrt{t}}$$

$$u_{xx} = c\, e^{-\frac{x^2}{4t}} \cdot \left(-\frac{2x}{4t}\right)\left(\frac{1}{\sqrt{t}}\right) = -\frac{c}{2} x t^{-3/2} e^{-\frac{x^2}{4t}}$$

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Another solution, fundamental solution.

$$u_t = u_{xx}$$

with $u(t,x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$, $t > 0$, $x \in \mathbb{R}$.