Lecture 1

$\mathcal{P}(\mathbb{R}^n)$ is the set of all subsets of $\mathbb{R}^n$.

$\mu : \mathcal{P}(\mathbb{R}^n) \to [0,\infty]$ is an outer measure if

(i) $\mu(\emptyset) = 0$

(ii) $E \subset \bigcup_{h\in\mathbb{N}} E_h \Rightarrow \mu(E) \leq \sum_{h\in\mathbb{N}} \mu(E_h)$

(ii) is called σ- subadditivity and implies monotonicity of μ:

$E \subset F \Rightarrow \mu(E) \leq \mu(F)$

<u>Examples of outer measures:</u>

<u>Ex 1</u>: The Dirac measure δ_x at $x \in \mathbb{R}^n$,

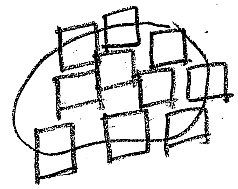
$$\delta_x(E) = \begin{cases} 1, & x \in E \\ 0, & x \notin E \end{cases}$$

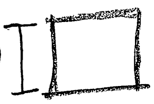
<u>Ex 2</u>: The counting measure $\#$:

$$\#(E) = \begin{cases} \text{number of elements of } E, & E \text{ is finite} \\ +\infty, & \text{if } E \text{ is infinity} \end{cases}$$

Ex 3: The Lebesgue measure $\mathcal{L}^n$:

$$\mathcal{L}^n(E) = \inf_{\mathcal{F}} \sum_{Q \in \mathcal{F}} r(Q)^n, \qquad E \subset \mathbb{R}^n,$$

E 

$r(Q)$ 

$\mathcal{F}$ is a countable covering of E by cubes with sides parallel to the coordinate axis.

$r(Q)$ = side length of Q.

Notation: $\mathcal{L}^n(E) = |E|$, the volume of E.

$$|x + E| = |E|, \quad x \in \mathbb{R}^n, \qquad |\lambda E| = \lambda^n |E|, \quad \lambda > 0.$$

Def: Hausdorff measure.

Fix n, consider $\mathbb{R}^n$. In $\mathbb{R}^n$ we can define, for each $0 \leq s \leq n$ and $\delta > 0$, the following outer measures:

$$\boxed{H^s_\delta(E) = \inf \left\{ \sum_{j=1}^{\infty} w_s \left(\frac{\operatorname{diam} E_j}{2}\right)^s : E \subset \bigcup_{j=1}^{\infty} E_j, \ \operatorname{diam} E_j < \delta \right\}}$$

where:

$$w_s = \frac{\pi^{s/2}}{\Gamma(1 + s/2)}, \qquad s \geq 0$$

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} \, dt, \qquad s > 0.$$

For $K = 1, 2, 3, \dots$

$$\omega_K = \pi^{K/2}\,\Gamma(1+K/2)^{-1},$$

where,

$$\omega_K = |\{x \in \mathbb{R}^K : |x| < 1\}|,$$

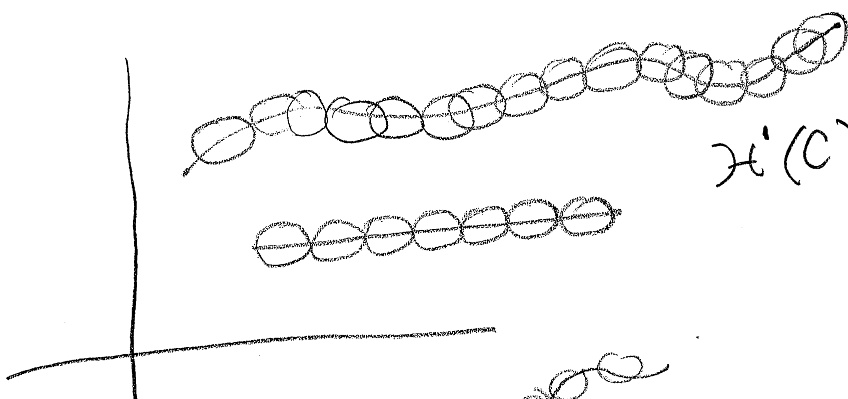
So:

$$\omega_1 = 2, \quad \omega_2 = \pi, \quad \omega_3 = \frac{4}{3}\pi, \dots$$

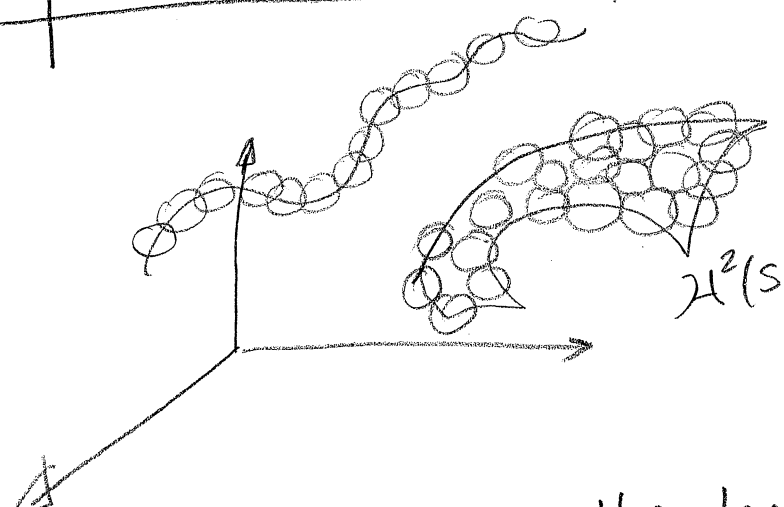
Moreover, we define the <u>outer measure</u>:

$$\mathcal{H}^s(E) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(E) = \sup_{\delta > 0} \mathcal{H}^k_\delta(E)$$

$n=2$



$\mathcal{H}^1(C) =$ length of C.



$n=3$

$\mathcal{H}^2(S) =$ Area of surface S.

We want to measure the length of curves or the area of surfaces without using parametric representations.

Remarks

* Given n, we have the measures $\mathcal{H}^s$, $0 \leq s \leq n$. It can be proven that $\mathcal{H}^n = \mathcal{L}^n$. Moreover $\mathcal{H}^0$ is the counting measure $\#$. Also $\mathcal{H}^s \equiv 0$, $s > n$.

* In the definition of $\mathcal{H}^s_\delta$, the covering can be made using:

  E_j closed sets,
  E_j open sets,
  E_j convex sets,
  E_j so that $E_j \subset E$.
  E_j that intersect E only.

Def: . μ is concentrated on E if $\mu(\mathbb{R}^n \setminus E) = 0$

. $Supp(\mu) = \{x \mid \mu(B_r(x)) > 0, \forall r > 0\}$

$Supp(\mu)$ is a closed set.

μ can be concentrated on sets smaller than $supp(\mu)$.

$$supp(\mu) = \{\cap E : E \text{ is closed}, \mu(\mathbb{R}^n \setminus E) = 0\}$$

Ex: Let $\{x_h\}_{h \in \mathbb{N}}$ be a sequence in $\mathbb{R}^n$. Let

$$\mu = \sum_{h=1}^{\infty} \frac{1}{2^h} \delta x_h$$

μ is concentrated on the elements of the sequence but supp μ contains every accumulation point of the sequence.

We would like to have that μ is σ-additive:

If $E = \bigcup_{j=1}^{\infty} E_j \Rightarrow \mu(E) = \sum_{j=1}^{\infty} \mu(E_j)$, $\{E_j\}$ disjoint.

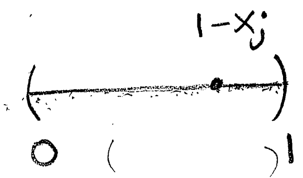
This fails for the Vitali set and $\mu = \mathcal{L}^1$:

Let $(0,1)$. Then $x \sim y$ if $x - y \in \mathbb{Q}$. Let

$$E := \{\text{one element from every class}\}$$

If $\{x_j\} = \mathbb{Q} \cap (0,1)$, define:

$$E_j = \left[x_j + (E \cap (0, 1 - x_j))\right] \cup \left[(x_j - 1) + (E \cap (1 - x_j, 1))\right]$$



$(x_j, 1)$ $(0, x_j - 1)$

$\{E_j\}$ are disjoint

$|E_j| = |E|$, $(0,1) = \bigcup_{j=1}^{\infty} E_j$

The σ-additivity of $\mathcal{L}^1$ on $\{E_j\}$ would then imply:

$$1 = |(0,1)| = \sum_{j=1}^{\infty} |E_j|,$$

which contradicts $|E_j| = |E| \in [0, \infty]$.

<u>Def</u>: E is μ-measurable ($E \in \mathcal{M}(\mu)$) if

$$\mu(F) = \mu(F \cap E) + \mu(F \setminus E), \quad \forall F.$$

(Remark: only need $\geq$, since $\mu(F) \leq \mu(F \cap E) + \mu(F \setminus E)$ is always true by subadditivity).

<u>Def</u>: $\mathcal{M} \subset \mathcal{P}(\mathbb{R}^n)$ is a σ-algebra on $\mathbb{R}^n$ if:

(1) $E \in \mathcal{M} \Rightarrow \mathbb{R}^n \setminus E \in \mathcal{M}$

(2) $\{E_j\} \subset \mathcal{M} \Rightarrow \bigcup_{j=1}^{\infty} E_j \in \mathcal{M}$

(3) $\mathbb{R}^n \in \mathcal{M}$.

<u>Def</u>: If $\mathcal{M}$ is a σ-algebra on $\mathbb{R}^n$, then $\mu : \mathcal{M} \to [0,\infty]$ is a measure on $\mathcal{M}$ if $\mu(\emptyset) = 0$ and if μ is σ-additive on $\mathcal{M}$.

<u>Carathéodory's theorem</u>: If μ is an outer measure on $\mathbb{R}^n$, and $\mathcal{M}(\mu)$ is the set of all measurable sets, then $\mathcal{M}(\mu)$ is a σ-algebra and μ is a measure on $\mathcal{M}(\mu)$.

<u>Corollary</u>: $E_i \nearrow$ $(E_{i+1} \supseteq E_i)$, $E_i \in \mathcal{M}(\mu) \Rightarrow$

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{i \to \infty} \mu(E_i)$$

$E \downarrow$, $(E_{i+1} \subseteq E_i)$, $\mu(E_1) < \infty$, $\Rightarrow$

$$\mu\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{i \to \infty} \mu(E_i)$$

<u>Def</u>: μ is σ-finite if $\mathbb{R}^n = \bigcup_{i=1}^{\infty} E_i$,
$E_i \in \mathcal{M}(\mu)$, $\mu(E_i) < \infty$.

<u>Integration</u>:
$u: \mathbb{R}^n \to [-\infty, \infty]$ is μ-measurable if
$\{u > t\} \in \mathcal{M}(\mu)$ $\forall t \in \mathbb{R}$.

u is simple if $u(\mathbb{R}^n)$ is countable.

<u>Def</u>: • $u: \mathbb{R}^n \to [0, \infty]$, μ-meas., simple

$$\int u \, d\mu = \sum_{t \in u(\mathbb{R}^n)} t \, \mu(u^{-1}(t))$$

• $u: \mathbb{R}^n \to [-\infty, \infty]$, simple and $\int u^+ d\mu < \infty$ or $\int u^- d\mu < \infty$, then u is μ-integrable and

$$\int u \, d\mu = \int u^+ d\mu - \int u^- d\mu$$

• $u: \mathbb{R}^n \to [-\infty, \infty]$, μ-meas, define:

$$\int^*_{\mathbb{R}^n} u \, d\mu = \inf \left\{ \int_{\mathbb{R}^n} v \, d\mu : v \geq u \ \mu \text{ a.e}, v \text{ simple} \right\}$$

$$\int_{*\mathbb{R}^n} u \, d\mu = \sup \left\{ \int_{\mathbb{R}^n} v \, d\mu : v \leq u \ \mu \text{ a.e}, v \text{ simple} \right\}$$

If upper and lower integrals coincide then u is integrable and:

$$\int_{\mathbb{R}^n} u \, d\mu = \int^*_{\mathbb{R}^n} u \, d\mu = \int_{*\mathbb{R}^n} u \, d\mu.$$

<u>Def</u>: u is μ-summable if:

$$\int_{\mathbb{R}^n} |u| \, d\mu < \infty,$$

and we write $u \in L^1(\mathbb{R}^n, \mu)$

<u>Def</u>: $\int_E u \, d\mu = \int_{\mathbb{R}^n} \chi_E u \, d\mu$

$$⨍_E u \, d\mu = \frac{1}{\mu(E)} \int_E u \, d\mu$$

<u>Egoroff's theorem</u>: If $\{u_j\}$ is a sequence of μ-measurable functions, $u_j \to u$ pointwise, then $\forall \varepsilon > 0$ and $\forall E \in \mathcal{M}(\mu)$, $\mu(E) < \infty$, $\exists F \in \mathcal{M}(\mu)$ s.t.:

$\mu(E \setminus F) < \varepsilon$, $u_j \to u$ uniformly on F.

<u>Product measure</u>: μ outer measure on $\mathbb{R}^n$, ν outer measure on $\mathbb{R}^m$. Define outer measure:

$$\mu \times \nu : \mathcal{P}(\mathbb{R}^n \times \mathbb{R}^m) \to [0, \infty]$$

$$\mu \times \nu (G) = \inf_{\mathcal{F}} \sum_{E \times F \in \mathcal{F}} \mu(E) \nu(F),$$

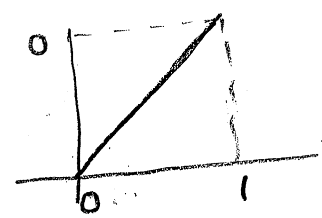
where $\mathcal{F}$ is a <u>countable</u> covering of G by sets of the form $E \times F$, $E \in \mathcal{M}(\mu)$, $F \in \mathcal{M}(\nu)$

Fubini's Theorem :

$$u \in L^1(\mu\times\nu) \Rightarrow \begin{cases} \displaystyle\int_{\mathbb{R}^n\times\mathbb{R}^m} u \, d(\mu\times\nu) = \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^m} u(x,y)\, d\nu(y)\right) d\mu(x) \\ \displaystyle\qquad = \int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^n} u(x,y)\, d\mu(x)\right) d\nu(y) \end{cases}$$

Ex: This example shows that Fubini's theorem might not be true if $u \notin L^1(\mu\times\nu)$:

Let $u(x,y) = \chi_{\{x=y,\, 0\le x\le 1\}}$



with $\mu = \mathcal{L}^1$ and $\nu = \#$, $u \notin L^1(\mu\times\nu)$.

$$\int_0^1 \Big[\underbrace{\int_{\mathbb{R}} u(x,y)\, d\#(y)}_{=1}\Big] dx = 1$$

$$\int_0^1 \Big[\underbrace{\int_0^1 u(x,y)\, dx}_{0}\Big] d\#(y) = 0$$

__Corollary:__ $u \in L^1(\mu)$, $u \ge 0 \Rightarrow \int_{\mathbb{R}^n} u \, d\mu = \int_0^\infty \mu(\{u>t\})\, dt$

$\int |u|^p d\mu = p\int_0^\infty t^{p-1}\mu(\{|u|>t\})\, dt$. (Apply Fubini's thm on $\mu\times dt$ and $f(x,t) = \chi_{(0,u(x))}(t)$)