Lecture 10

Blow-ups of Radon measures and rectifiability.

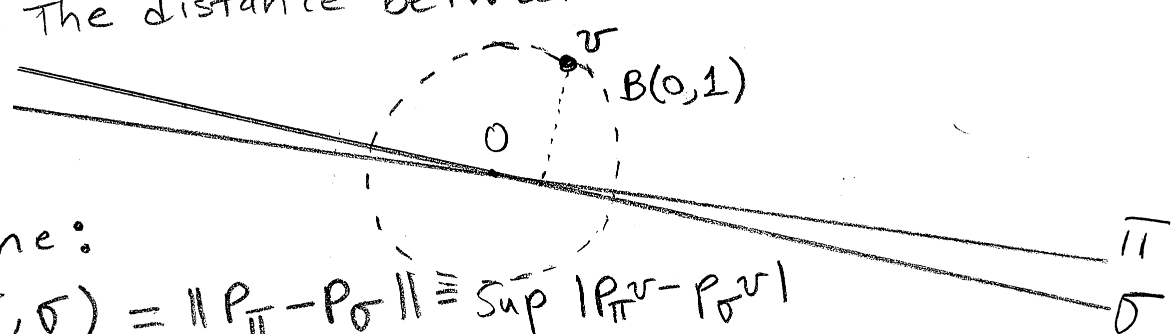
We now prove the converse of Theorem 2 (see Lecture 8 and 9) which is going to be used to study the structure of sets of finite perimeter.

<u>Theorem 3</u>: (Rectifiability from convergence of the blow-ups). Let μ be Radon measure on $\mathbb{R}^n$, μ concentrated on M (i.e, M Borel set in $\mathbb{R}^n$, $\mu(\mathbb{R}^n \setminus M) = 0$), and, for every $x \in M$, $\exists \Pi_x$ a k-dimensional plane Π_x in $\mathbb{R}^n$ such that:

$$\frac{(\Phi_{x,r})_{\#}\mu}{r^k} \xrightarrow{*} \mathcal{H}^k \llcorner \Pi_x$$

as $r \to 0^+$, then $\mu = \mathcal{H}^k \llcorner M$ and M is locally $\mathcal{H}^k$-rectifiable.

<u>Proof</u>: The distance between two k-dim planes in $\mathbb{R}^n$:



Define:

$$d(\Pi, \sigma) = \| P_\Pi - P_\sigma \| \equiv \sup_{|v|=1} |P_\Pi v - P_\sigma v|$$

$$d(\Pi, \sigma) = 0 \iff \Pi = \sigma$$

We will use the following:

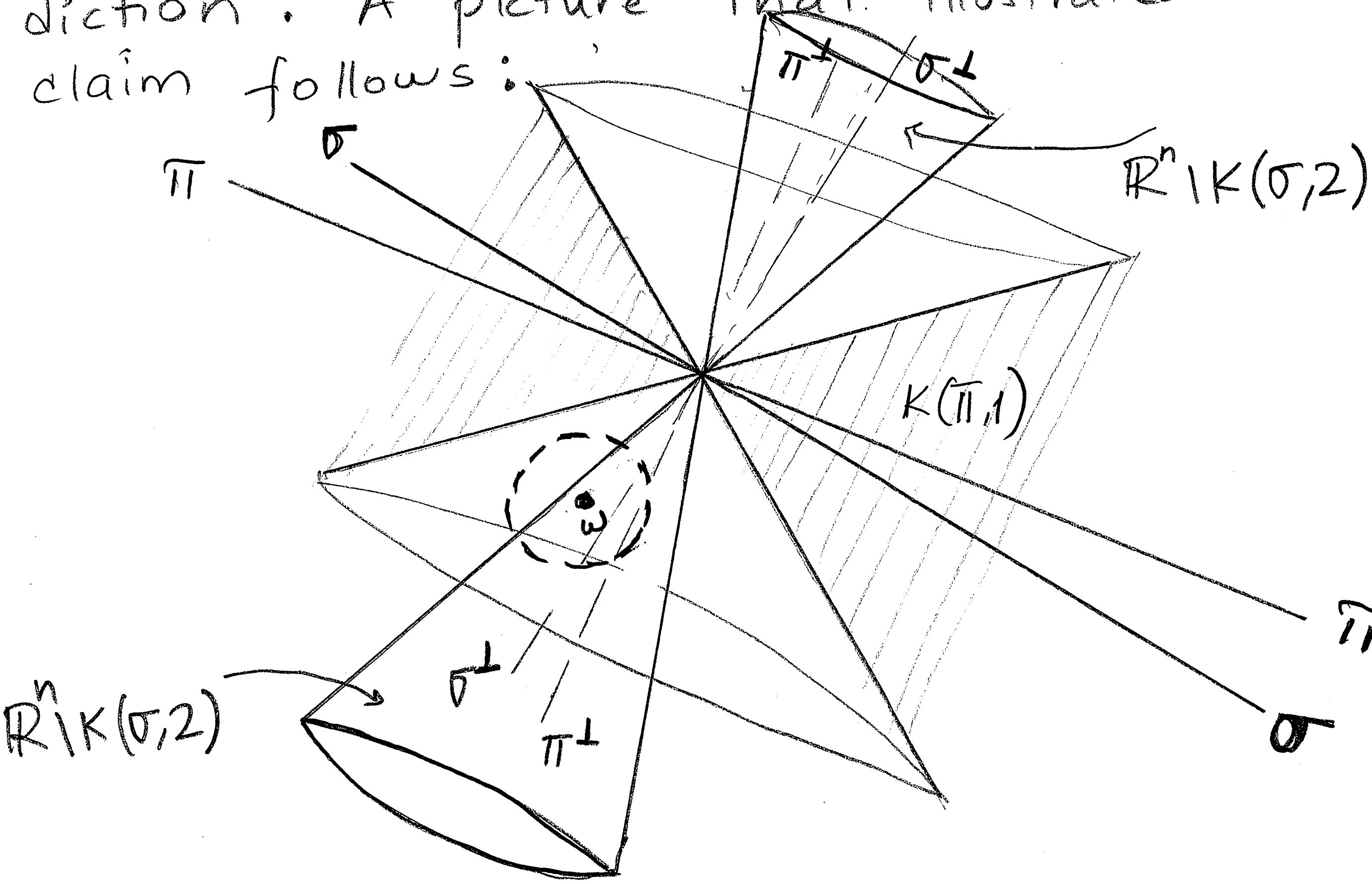
<u>Claim</u>: $\exists\, \lambda \in (0,1)$ such that, if π, σ are k-dim. planes in $\mathbb{R}^n$ with

$$d(\pi, \sigma) < \lambda$$

then

$$B(w, \lambda|w|) \cap K(\pi, 1) = \emptyset, \quad \forall w \in \mathbb{R}^n \setminus K(\sigma, 2).$$

This claim can be proved by contradiction. A picture that illustrates this claim follows:



Recall:

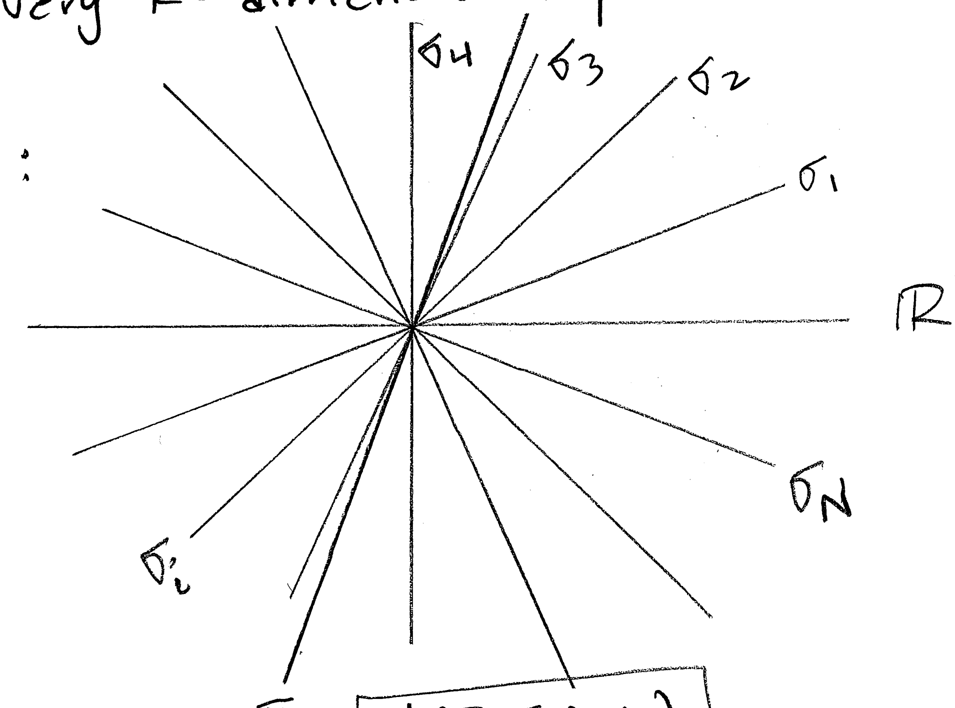
$$K(\pi, t) = \{x \in \mathbb{R}^n : |P_\pi^\perp x| \le t\, |P_\pi y|\}$$

Now, we fix a finite family of K-dimensional planes $\{\sigma_i\}_{i=1}^{N}$ with the property that:

$$\min_{1\le i\le N} d(\sigma_i, \Pi) < \lambda$$

for every K-dimensional plane Π in $\mathbb{R}^n$.

Ex :



$\boxed{d(\Pi, \sigma_3) < \lambda}$.

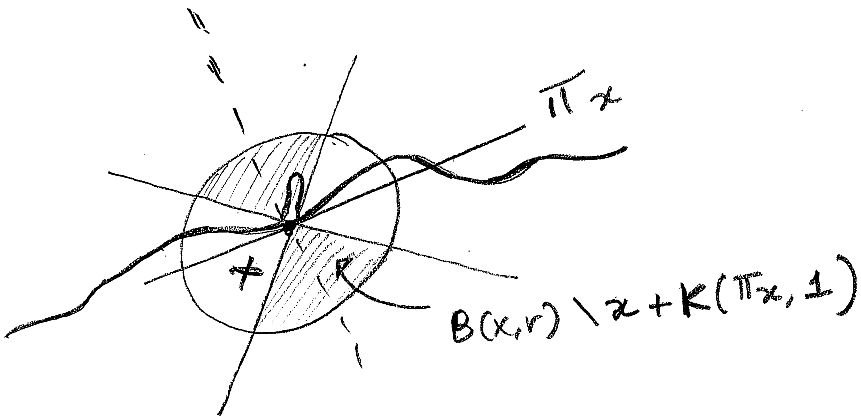
The proof of Theorem 3 has now 3 steps:

Step 1. : We show that if $M' \subset M$, M' compact and

$$\lim_{r\to 0^+} \frac{\mu(B(x,r))}{\omega_K r^K} = 1$$

$$\lim_{r\to 0^+} \frac{\mu(B(x,r)\setminus(x + K(\Pi_x, 1)))}{\omega_K r^K} = 0$$

UNIFORMLY with respect to $x \in M'$, then M' is $\mathcal{H}^K$-rectifiable.



Uniform convergence $\Rightarrow \forall \varepsilon>0, \exists \delta>0$ s.t.

$$(A)\quad \boxed{\begin{array}{l}\mu(B(x,r)) \geq (1-\varepsilon)\omega_k r^k \\ \mu(B(x,r)) \setminus (x+K(\pi_x,1))) \leq \varepsilon \omega_k r^k\end{array}} \quad \begin{array}{l}\forall x \in M' \\ r \leq 2\delta\end{array}$$

We can decompose:

$$M' = \bigcup_{i=1}^{N} M_i', \text{ where}$$

$$M_i' = \{x \in M' : \text{dist}(\pi_x, \sigma_i) < \lambda\}$$

We will prove the each M_i' is $\mathcal{H}^k$-rectifiable, by using the Rectififiability criterion proved in Lecture 9. Indeed, by choosing $\varepsilon(k,\lambda)$ small enough we have:

$$\boxed{B(x,\delta) \cap M_i' \subset x + K(\sigma_i, 2), \quad \forall x \in M_i'}$$

because, if $y \in B(x,\delta) \cap M_i'$ but $y-x \in \mathbb{R}^n \setminus \{x+K(\sigma_i,2)\}$ then by claim in page 10.2

$$B(y, \lambda|y-x|) \subset \mathbb{R}^n \setminus \{x+K(\pi_x, 1))$$

Note that:

$\lambda < 1 \Rightarrow$

$$B(y, \lambda|y-x|) \subset B(x, 2|y-x|)$$

and hence

$$B(y, \lambda|y-x|) \subset B(x, 2|y-x|) \setminus (x + K(\Pi_x, 1))$$

$\Rightarrow$

$$\mu(B(y, \lambda|y-x|)) \leq \mu(B(x, 2|y-x|) \setminus (x + K(\Pi_x, 1))$$

$$\mu(B(y,\lambda|y-x|)) \geq (1-\varepsilon)\omega_K \lambda^K |y-x|^K; \text{ by (A)}$$

$$\mu(B(x, 2|y-x|) \setminus (x+K(\Pi_x,1))) \leq \varepsilon \omega_K 2^K |y-x|^K \ ; \text{ by (A)}$$

By we see a contradiction for $\varepsilon(K, \lambda)$ small enough (in particular for $\varepsilon < \lambda^K/(2^K+\lambda^K)$). Indeed:

$$(1-\varepsilon)\omega_K \lambda^K |y-x|^K \leq \varepsilon \, \omega_K 2^K |y-x|^K$$

$$\Leftrightarrow \quad \lambda^K - \varepsilon\lambda^K \leq \varepsilon 2^K$$

$$\Leftrightarrow \quad \lambda^K \leq \varepsilon(2^K + \lambda^K)$$

$$\Leftrightarrow \quad \varepsilon \geq \frac{\lambda^K}{2^K + \lambda^K}$$

We conclude that $B(x,\delta) \cap M_i' \subset x + K(\sigma_i, 2)$, $\forall x \in M_i'$ and thus the <u>Rectifiability criteria</u> implies that M_i' is $\mathcal{H}^K$-rectifiable. Since each M_i' is $\mathcal{H}^K$-rectifiable then $M' = \bigcup_{i=1}^N M_i'$ is $\mathcal{H}^K$-rectifiable.

Step 2: We now prove that M is countably $\mathcal{H}^k$- rectifiable. First, recall that our hypothesis in Theorem 3 is:

$$\forall x \in M \Rightarrow \frac{(\Phi_{x,r})_\# \mu}{r^k} \xrightarrow{*} \mathcal{H}^k \llcorner \Pi_x .$$

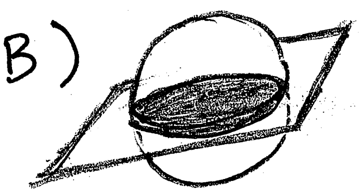
Take $B = B(x,1)$, for fixed $x \in M$. Then, clearly

$$(\mathcal{H}^k \llcorner \Pi_x)(\partial B) = 0$$

Then, by weak convergence of measures (Lecture 4, Page 4.10):

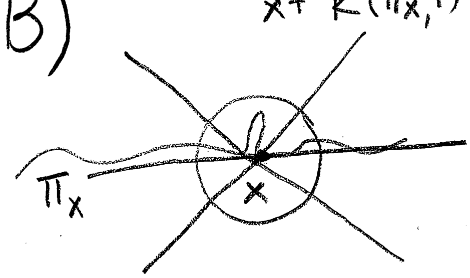
$$\lim_{r \to 0^+} \frac{(\Phi_{x,r})_\# \mu}{r^k}(B) = \mathcal{H}^k \llcorner \Pi_x (B)$$

$$\therefore \lim_{r \to 0^+} \frac{\mu(\Phi_{x,r}^{-1}(B))}{r^k} = \underbrace{\mathcal{H}^k(\Pi_x \cap B)}_{\omega_k}$$



$$\therefore \lim_{r \to 0^+} \frac{\mu(B(x,r))}{r^k} = \omega_k$$

$$\therefore \boxed{\lim_{r \to 0^+} \frac{\mu(B(x,r))}{\omega_k r^k} = 1} \quad (B)$$

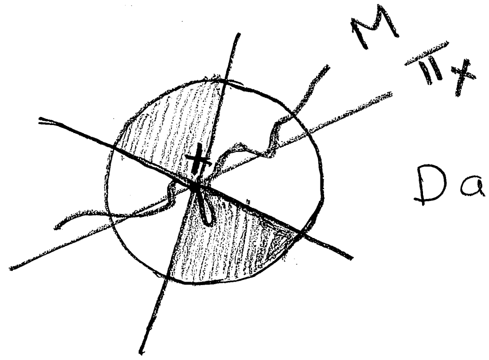


We do the same for

$$B(x,1) \setminus (x + K(\Pi_x, 1))$$

Note:

$$(\mathcal{H}^k \llcorner \Pi_x)\left(\partial\left(B(x,1) \setminus (x + K(\Pi_x,1))\right)\right) = 0$$



Dark area is
$B(x,1) \setminus (x + K(\Pi_x,1))$

By weak convergence.

$$\frac{(\Phi_{x,r})_{\#}\mu}{r^k}\left(B(x,1)\setminus(x+K(\Pi_x,1))\right) \xrightarrow[r\to 0]{} \mathcal{H}^k \llcorner \Pi_x \underbrace{\left(B(x,1)\setminus(x+K(\Pi_x,1))\right)}_{\overset{\shortparallel}{0}}$$

$$\therefore \quad \frac{\mu\left(\Phi_{x,r}^{-1}\left(B(x,1)\setminus(x+K(\Pi_x,1))\right)\right)}{r^k} \longrightarrow 0$$

Note, $\Phi_{x,r}^{-1}\left(B(x,1)\setminus(x+K(\Pi_x,1))\right) = B(x,r) \cap (x+K(\Pi_x,1))$

note that if we blow-up a cone we get again the same cone.

$$\Rightarrow \quad \boxed{\frac{\mu(B(x,r) \cap (x+K(\Pi_x,1)))}{\omega_k r^k} \xrightarrow[r\to 0]{} 0} \quad (C)$$

In order to apply Step 1, we need the convergence in (B) and (C) to be <u>UNIFORM</u>. We can accomplish this by applying Egoroff's theorem and regularity of measures (Lecture 2), Fix $R>0$:

Indeed, for each $i=1,2,\ldots$, $\exists M_i' \subset M \cap B_R$, M_i' compact such that:

- Limits in (B) and (C) are uniform $\forall x \in M'$
- $\mu((M \cap B_R) \setminus M_i') < \frac{1}{2^i}$, Let $E_i := (M \cap B_R) \setminus M_i'$

By Step 1,

$$M_i' \text{ is } \mathcal{H}^k\text{-rectifiably}, \quad i=1,2,\ldots$$

Now:

$$\mu\left((M \cap B_R) \setminus \bigcup_{i=1}^{\infty} M_i'\right) = 0; \text{ Use Borel Cantelli } \left(\mu\left(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} E_i\right) = 0\right)$$

But, in Step 3 we will see that

$$\mathcal{H}^k(E) \le \mu(E) \le 2^k \mathcal{H}^k(E) \quad \forall E \subset M \text{ Borel}$$

Thus:

$$\mathcal{H}^k\left((M \cap B_R) \setminus \bigcup_{i=1}^{\infty} M_i'\right) = 0,$$

Since $\bigcup_{i=1}^{\infty} M_i' \subset M \cap B_R$ is countably $\mathcal{H}^k$-rectifiable we conclude that $M \cap B_R$ is countably $\mathcal{H}^k$-rectifiable.

Since R is arbitrary $\Rightarrow M$ is countably $\mathcal{H}^k$-rectifiable.

Step 3: From Lecture 7 (Page 7.2)

we have:

$$\mathcal{H}^k(E) \le \mu(E) \le 2^k \mathcal{H}^k(E) \text{ ; from (B).}$$

$\therefore$ $\mathcal{H}^k(M \cap K) < \infty$ $\forall K \subset \mathbb{R}^n$ compact
(μ is Radon)

$\therefore$ M is locally $\mathcal{H}^k$-rectifiable

$\therefore$ $\mathcal{H}^k \llcorner M$ is Radon,
and $\mu << \mathcal{H}^k \llcorner M$ (since $\mu(E) \le 2^k \mathcal{H}^k(E)$)

By differentiation of measures:

$$\begin{cases} \theta(x) = \lim_{r \to 0^+} \dfrac{\mu(B(x,r))}{\mathcal{H}^k(M \cap B(x,r))} \text{ exists for } \mathcal{H}^k\text{-a.e. } x \in M. \\ \\ \mu = \theta \mathcal{H}^k \llcorner M \text{ on } \mathcal{B}(\mathbb{R}^n). \end{cases}$$

Now, M locally $\mathcal{H}^k$-rectifiably, by Thm 2 $\Rightarrow \theta = 1$ $\mathcal{H}^k$-a.e. $x \in M$

Because,

$$1 = \lim_{r \to 0} \frac{\mu(B(x,r))}{\omega_k r^k} = \lim_{r \to 0} \frac{\theta(x) \mathcal{H}^k(M \cap B(x,r))}{\omega_k r^k} = \theta(x) \lim_{r \to 0} \frac{\mathcal{H}^k(M \cap B(x,r))}{\omega_k r^k}$$

$= \theta(x) \cdot 1$; by Thm 2

$= \theta(x)$; for $\mathcal{H}^k$-a.e. $x \in M$

Finally, we have shown

$$\mu = \mathcal{H}^k \llcorner M \quad \text{on } \mathcal{B}(\mathbb{R}^n)$$

By Exercise 2.6, two Borel regular measures on $\mathbb{R}^n$ that agree on the Borel sets, are actually equal on all sets of $\mathbb{R}^n$.

$$\therefore \boxed{\mu = \mathcal{H}^k \llcorner M \quad \text{on } \mathcal{P}(\mathbb{R}^n)}$$