Lecture 12

Sets of finite perimeter

If E is a set with C^1-boundary, the following Gauss-Green theorem is proved in chapter 9 of the textbook:

Theorem 1: (Gauss-Green theorem) If E is an open set with C^1-boundary, then:

$$\int_E \nabla \varphi(x)\,dx = \int_{\partial E} \varphi \cdot \nu_E \, d\mathcal{H}^{n-1} \quad \forall \varphi \in C_c^1(\mathbb{R}^n)$$

where ν_E is the exterior unit normal to E.

Equivalently, the divergence theorem holds true:

$$\int_E \operatorname{div} T(x)\,dx = \int_{\partial E} T \cdot \nu_E \, d\mathcal{H}^{n-1}, \quad \forall T \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$$

This theorem is the starting motivation to study sets of finite perimeter. Indeed, we say that a Lebesgue measurable set $E \subset \mathbb{R}^n$ is a set of locally finite perimeter if $\exists \mu_E$, a Radon measure on $\mathbb{R}^n$ such that:

$$\int_E \nabla \varphi(x)\,dx = \int_{\mathbb{R}^n} \varphi \, d\mu_E, \quad \forall \varphi \in C_c^1(\mathbb{R}^n)$$

The total variation measure $|\mu_E|$ of μ_E induces the notion of relative perimeter $P(E;F)$ of E with respect to a set $F \subset \mathbb{R}^n$, and of total perimeter $P(E)$ of E, as:

$$P(E;F) = |\mu_E|(F), \quad P(E) = |\mu_E|(\mathbb{R}^n),$$

In particular, E is a set of finite perimeter if and only if $P(E) < \infty$.

For example, is E is an open set with C^1 boundary with outer unit normal $\nu_E \in C^1(\partial E, S^{n-1})$ then from the Gauss-Green formula in Theorem 1 it follows that E is a set of locally finite perimeter with:

$$\mu_E = \nu_E \mathcal{H}^{n-1} \llcorner \partial E, \qquad |\mu_E| = \mathcal{H}^{n-1} \llcorner \partial E,$$

$$P(E;F) = \mathcal{H}^{n-1}(F \cap \partial E), \quad P(E) = \mathcal{H}^{n-1}(\partial E).$$

Therefore, in the next six or seven lectures we will show that these definitions lead to a geometrically meaningful generalization of the notion of open set with C^1-boundary, with natural and powerful applications to the study of geometric variational problems.

Since, given $E \subset \mathbb{R}^n$ Lebesgue measurable, we want to produce a measure μ_E such that the Gauss-Green theorem holds, then it is natural to think of the Riesz representation theorem as the main analytical tool to produce μ_E. Thus, with the Riesz theorem in mind, we can now define:

<u>Definition</u>: Let $E \subset \mathbb{R}^n$ Lebesgue measurable. We say that E is a set of locally finite perimeter in $\mathbb{R}^n$ if for every compact set $K \subset \mathbb{R}^n$ we have:

$$\sup\left\{ \int_E \operatorname{div} T(x)\,dx : T \in C_c^1(\mathbb{R}^n;\mathbb{R}^n),\ \operatorname{spt} T \subset K,\ \sup_{\mathbb{R}^n} |T| \le 1 \right\} < \infty$$

If this quantity is bounded independently of K, then we say that E is a set of finite perimeter in $\mathbb{R}^n$.

With this definition and using the Riesz representation theorem we can prove the Gauss-Green formula for E.

<u>Theorem 2</u>: Let $E \subset \mathbb{R}^n$ Lebesgue measurable. Then E is a set of locally finite perimeter $\iff \exists\, \mu_E$, a $\mathbb{R}^n$-valued Radon measure on $\mathbb{R}^n$ such that:

$$\int_E \operatorname{div} T = \int_{\mathbb{R}^n} T \cdot d\mu_E \qquad \forall\, T \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$$

Moreover, E is a set of finite perimeter $\iff |\mu_E|(\mathbb{R}^n) < \infty$

Remark 1: Note that:

$\int_E \operatorname{div} T = \int_{\mathbb{R}^n} T \cdot d\mu_E$ is equivalent to $\int_E \nabla\varphi = \int_{\mathbb{R}^n} \varphi \, d\mu_E$, $\varphi \in C_c^1(\mathbb{R}^n)$
$T \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$

Indeed:

Let $\varphi \in C_c^1(\mathbb{R}^n)$. Let $T_i := (0, \dots, \varphi, \dots 0)$ ← i position.

$$\Rightarrow \int_E \operatorname{div} T_i = \int_{\mathbb{R}^n} T_i \cdot d\mu_E$$

$$\therefore \int_E \varphi_{x_i} = \int_{\mathbb{R}^n} \varphi \, (d\mu_E)_i \quad , \quad i = 1, 2, \dots, n$$

Since $\nabla\varphi = (\varphi_{x_1}, \dots, \varphi_{x_n}) \Rightarrow$

$$\int_E \nabla\varphi = \int_{\mathbb{R}^n} \varphi \, d\mu_E$$

On the other hand:

Let $T \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$, $T = (\varphi_1, \dots, \varphi_n)$. Since:

$$\int_E \nabla\varphi_i = \int_{\mathbb{R}^n} \varphi_i \, d\mu_E \Rightarrow \int_E (\varphi_i)_{x_i} = \int_{\mathbb{R}^n} \varphi_i \, (d\mu_E)_i ,$$

for $i = 1, 2, \dots, n$, Hence:

$$\sum_{i=1}^n \int_E (\varphi_i)_{x_i} = \sum_{i=1}^n \int_{\mathbb{R}^n} \varphi_i \, (d\mu_E)_i$$

$$\therefore \int_E \operatorname{div} T = \int_{\mathbb{R}^n} T \cdot d\mu_E$$

Definition: We call μ_E the Gauss–Green measure of E, and define the relative perimeter of E in $F \subset \mathbb{R}^n$, and the perimeter of E, as:

$$P(E;F) = |\mu_E|(F), \quad P(E) = |\mu_E|(\mathbb{R}^n).$$

Proof of Theorem 2:

Let E be a set of locally finite perimeter.
Define:

$$L : C_c^1(\mathbb{R}^n;\mathbb{R}^n) \to \mathbb{R}$$

$$\langle L, T\rangle = \int_E \operatorname{div} T(x)\,dx.$$

Let $K \subset \mathbb{R}^n$ compact;

$$\sup\{\langle L,T\rangle : T \in C_c^1(\mathbb{R}^n;\mathbb{R}^n), \operatorname{spt} T \subset K, |T| \le 1\} < \infty$$

because, by definition of set of finite perimeter:

$$|\langle L,T\rangle| = \left|\int_E \operatorname{div} T(x)\,dx\right| \le C(K), \quad \forall T \in C_c^1(\mathbb{R}^n;\mathbb{R}^m), \ |T| \le 1.$$

Hence, L is continuous in $C_c^1(\mathbb{R}^n;\mathbb{R}^n)$, with respect to the topology in $C_c(\mathbb{R}^n;\mathbb{R}^n)$ introduced in Lecture 4. Hence, L can be extended by density to a bounded continuous linear functional on $C_c(\mathbb{R}^n;\mathbb{R}^n)$. By Riesz's theorem $\exists \mu_E$ such that:

$$\langle L,T\rangle = \int_{\mathbb{R}^n} T\cdot d\mu_E \implies \int_E \operatorname{div} T(x)\,dx = \int_{\mathbb{R}^n} T\cdot d\mu_E$$

The converse implication is trivial. ∎

Remark 2: Let E be a set of locally finite perimeter in $\mathbb{R}^n$. If $|E \Delta F| = 0$, then:

$$\int_E \operatorname{div} T = \int_F \operatorname{div} T \, , \quad \int_E \operatorname{div} T = \int_{\mathbb{R}^n} T \cdot d\mu_E \quad \forall T \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$$

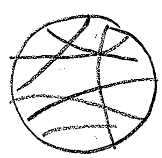
Thus, $\int_F \operatorname{div} T = \int_{\mathbb{R}^n} T \cdot d\mu_E \;\; \forall T \in C_c^1(\mathbb{R}^n; \mathbb{R}^n) \Rightarrow F$ is of locally finite perimeter

Hence, $\exists \mu_F$ s.t $\int_F \operatorname{div} F = \int_{\mathbb{R}^n} T \cdot d\mu_F \;\; \forall T \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$

In view of Remark 1, and since $C_c^1(\mathbb{R}^n)$ is dense in $C_c(\mathbb{R}^n)$ (See Lecture 4, page 4.5).

$$\Rightarrow \int_{\mathbb{R}^n} \varphi \, d\mu_E = \int_{\mathbb{R}^n} \varphi \, d\mu_F \;\; \forall \varphi \in C_c(\mathbb{R}^n) \Rightarrow \mu_E = \mu_F$$

We can actually modify E in such a way that the new set F has a "huge" topological boundary but still $\mu_F = \mu_E$. For example, let $E \subset \mathbb{R}^2$ be the unit disk and $F = E \cup \mathbb{Q}^2$. Thus, $|E \Delta F| = 0$, but $\mu_F = \nu_E \mathcal{H}^1 \llcorner \partial E$. Or F can be as follows:

$E =$  $F =$  disk E minus all the curves in the picture

$$|E \Delta F| = 0$$

Actually, By the Gauss-Green theorem, note that if $E \subset \mathbb{R}^n$ is open (not necessarily bounded) with C^1 boundary, then E is a set of locally finite perimeter with $\mu_E = \nu_E \mathcal{H}^{n-1} \llcorner \partial E$, $P(E) = \mathcal{H}^{n-1}(\partial E)$, and $P(E, F) = \mathcal{H}^{n-1}(\partial E \cap F) \;\; \forall F \subset \mathbb{R}^n$.

In chapter 9 of the textbook, the Gauss-Green formula is proved to be true also for sets E with Lipschitz boundary or polyhedral boundary. Hence such sets E are of locally finite perimeter, with $P(E;F) = \mathcal{H}^{n-1}(F \cap \partial E)$ whenever $F \subset \mathbb{R}^n$. Moreover, if E is bounded, then E is of finite perimeter.

Since convex sets have locally Lipschitz boundary, it follows that convex sets are of locally finite perimeter, while bounded convex sets are of finite perimeter.

Remark 3: Recall from the theory of distributions that if $f \in L^1_{loc}(\mathbb{R}^n)$, then f induces a distribution T_f defined as $\langle T_f, \varphi \rangle = \int_{\mathbb{R}^n} f\varphi \, dx$.
Moreover, the derivative of the distribution T_f is:

$$\langle DT_f, \varphi \rangle = -\int_{\mathbb{R}^n} f \nabla \varphi, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^n).$$

Then, if $E \subset \mathbb{R}^n$ is Lebesgue measurable $\Rightarrow \chi_E \in L^1_{loc}(\mathbb{R}^n)$
Hence:
$E \subset \mathbb{R}^n$ is a set of locally finite perimeter $\Longleftrightarrow$ the distributional gradient $D\chi_E$ can be represented as the integration with respect to $-\mu_E$.

Lower semicontinuity of perimeter.

We say that E_i locally converges to E ($E_i \xrightarrow{loc} E$) if

$$\chi_{E_i} \to \chi_E \quad \text{in } L^1_{loc}(\mathbb{R}^n)$$

That is,

$$\lim_{i\to\infty} |K \cap (E_i \Delta E)| = 0, \quad \forall K \subset \mathbb{R}^n \text{ compact}$$

We simply say that E_i converges to E, $E_i \to E$, if $\chi_{E_i} \to \chi_E$ in $L^1(\mathbb{R}^n)$; that is:

$$\lim_{i\to\infty} |E \Delta E_i| = 0$$

Remark 4: Let E be a set of locally finite perimeter. Then, Theorem 2 implies $\exists \mu_E$ Radon s.t.

$$\int_E \operatorname{div} T(x)\, dx = \int_{\mathbb{R}^n} T \cdot d\mu_E \qquad \forall T \in C^1_c(\mathbb{R}^n; \mathbb{R}^n),$$

and by our study of the Riesz's theorem in Lecture 4 (See page 4.3) we have; for A open:

$$P(E,A) = |\mu_E|(A) = \sup \left\{ \int_E \operatorname{div} T(x)\, dx : T \in C^\infty_c(A;\mathbb{R}^n), |T| \leq 1 \right\}$$

Note: C^∞_c, C^1_c are both dense in C_c

Theorem 3 (Lower semicontinuity of perimeter):

Let $\{E_i\}$ sequence of sets of locally finite perimeter in $\mathbb{R}^n$, with

$$E_i \xrightarrow{loc} E; \quad \limsup_{i\to\infty} P(E_i; K) < \infty \quad \forall K \subset \mathbb{R}^n \text{ compact.}$$

Then:

(a) E is a set of locally finite perimeter in $\mathbb{R}^n$

(b) $\mu_{E_i} \xrightarrow{*} \mu_E$

(c) $P(E; A) \leq \liminf_{i\to\infty} P(E_i; A)$, $\forall A \subset \mathbb{R}^n$ open.

Proof:

By Remark 4; for $T \in C_c^1(A; \mathbb{R}^n)$, $|T| \leq 1$, A open:

$$\int_E \operatorname{div} T(x) = \lim_{i\to\infty} \int_{E_i} \operatorname{div} T(x)\, dx = \lim_{i\to\infty} \int_{\mathbb{R}^n} T \cdot d\mu_{E_i}$$

$$\leq \liminf_{i\to\infty} |\mu_{E_i}|(A)$$

$\therefore$ E is a set of locally finite perimeter (using $A = K$ and hypothesis)

$\therefore$ $\operatorname{Per}(E; A) \leq \liminf_{i\to\infty} P(E_i, A)$; even for A unbounded.

Now, since $E_i \xrightarrow{loc} E$, we have:

$$\int_{E_i} \nabla\varphi\, dx \to \int_E \nabla\varphi\, dx \quad \therefore \int_{\mathbb{R}^n} \varphi\, d\mu_{E_i} \to \int_{\mathbb{R}^n} \varphi\, d\mu_E \quad \forall \varphi \in C_c^1(\mathbb{R}^n)$$

Since $C_c^1(\mathbb{R}^n)$ is dense in $C_c(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} \varphi\, d\mu_{E_i} \to \int_{\mathbb{R}^n} \varphi\, d\mu_E, \quad \forall \varphi \in C_c(\mathbb{R}^n); \text{ i.e. } \mu_{E_i} \xrightarrow{*} \mu_E.$$

As explained in Remark 2, we can modify a set of locally finite perimeter E by a set of $\mathcal{L}^n$-measure zero without changing its Gauss–Green measure, and, as a consequence, its perimeter. Such modifications may largely increase the topological boundary. The following Lemma shows how to modify E to "minimize" the size of the topological boundary.

<u>Lemma 1</u>: If E is a set of locally finite perimeter in $\mathbb{R}^n$, then:

$$(*) \quad \operatorname{spt} \mu_E = \{x \in \mathbb{R}^n : 0 < |E \cap B(x,r)| < \omega_n r^n, \ \forall r > 0\} \subset \partial E$$

Moreover, there exists a Borel set F such that:

$$|E \Delta F| = 0, \quad \operatorname{spt} \mu_F = \partial F$$

<u>Proof</u>: If $x \in \mathbb{R}^n$, $|E \cap B(x,r)| = 0$, for some $r > 0$, then

$$\int_E \nabla \varphi \, dx = \int_{\mathbb{R}^n} \varphi \, d\mu_E, \quad \forall \varphi \in C_c^\infty(B(x,r))$$

$$\| $$

$$\int_{E \cap B(x,r)} \nabla \varphi \, dx = 0$$

Exercise 4.14

$$\therefore \int_{\mathbb{R}^n} \varphi \, d\mu_E = 0 \quad \forall \varphi \in C_c^\infty(B(x,r)) \Rightarrow |\mu_E|(B(x,r)) = 0$$

$$\Rightarrow x \notin \operatorname{spt} \mu_E$$

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If $x \in \mathbb{R}^n$, $|E \cap B(x,r)| = |B(x,r)|$ for some $r>0$, then; for $\varphi \in C_c^\infty(B(x,r))$:

$$\int_E \nabla \varphi = \int_{\mathbb{R}^n} \varphi \, d\mu_E$$

$$\| $$

$$\int_{E \cap B(x,r)} \nabla \varphi$$

$$\| $$

$$\int_{B(x,r)} \nabla \varphi = 0$$

Exercise 4.14

$\therefore \int_{\mathbb{R}^n} \varphi \, d\mu_E = 0 \quad \forall \varphi \in C_c^\infty(B(x,r)) \Rightarrow |\mu_E|(B(x,r)) = 0$

$\Rightarrow x \notin \operatorname{spt} \mu_E$

Also, if $x \notin \operatorname{spt} \mu_E \Rightarrow |\mu_E|(B(x,r)) = 0$, some $r>0$.
and, for $\varphi \in C_c^\infty(B(x,r))$:

$$0 = \int_{\mathbb{R}^n} \varphi \, d\mu_E = \int_E \nabla \varphi = \int_{\mathbb{R}^n} \chi_E \nabla \varphi .$$

By Lemma 7.5 in textbook ($u \in L^1_{loc}(\mathbb{R}^n)$, A open connected, $\int_{\mathbb{R}^n} u \nabla \varphi = 0$, $\forall \varphi \in C_c^\infty(A) \Rightarrow u = c \in \mathbb{R}$ a.e. in A) it follows that:

$$\chi_E = c \quad \text{a.e. on } B(x,r)$$

$$\Rightarrow |E \cap B(x,r)| \in \{0, \omega_n r^n\}.$$

$\therefore \operatorname{spt} \mu_E = \{x \in \mathbb{R}^n : 0 < |E \cap B(x,r)| < \omega_n r^n \ \forall r>0\} \subset \partial E$

To find F, WLOG E is Borel (by regularity of $\mathcal{L}^n$). Define:

$$A_0 := \{x \in \mathbb{R}^n : \exists r>0 \text{ s.t. } |E \cap B(x,r)| = 0\}$$

$$A_1 := \{x \in \mathbb{R}^n : \exists r>0 \text{ s.t. } |E \cap B(x,r)| = \omega_n r^n\}$$

Let $\{x_i\} \subset A_0$, $A_0 \subset \bigcup_{i=1}^{\infty} B(x_i, r_i)$, $r_i > 0$, $|E \cap B(x_i, r_i)| = 0$.

$$\Rightarrow |E \cap A_0| = 0$$

$$\Rightarrow |A_1 \setminus E| = 0 \quad ; \quad \text{since } \mu_{\mathbb{R}^n \setminus E} = -\mu_E$$

A_1 for E is A_0 for $\mathbb{R}^n \setminus E$.

Exercise 12.9 in textbook

Define Borel set:

$$\boxed{F := (A_1 \cup E) \setminus A_0}$$

With:

$$|F \setminus E| \le |A_1 \setminus E| = 0, \quad |E \setminus F| \le |E \cap A_0| = 0$$

$$\therefore \quad |E \Delta F| = 0.$$

By $(*)$:

$$\underline{\operatorname{spt} \mu_F} = \operatorname{spt} \mu_E = \mathbb{R}^n \setminus (A_0 \cup A_1) \underline{\subset \partial F}$$

On the other hand, $\partial F \subset \operatorname{spt} \mu_F$ because:

$A_1 \subset \mathring{F}$ (by construction), $\bar{F} \subset \mathbb{R}^n \setminus A_0$.

We conclude:

$$\boxed{\operatorname{spt} \mu_F = \partial F} \quad \blacksquare$$