Lecture 13

Compactness from perimeter bounds.

First, we stop to study the convolutions of characteristic functions of sets of locally finite perimeter with regularizing Kernels.

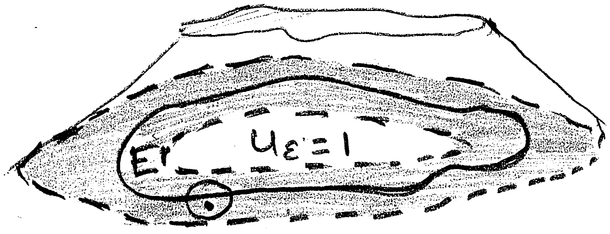
Let $E \subset \mathbb{R}^n$ be a Lebesgue measurable set. Then $\chi_E \in L^1_{loc}(\mathbb{R}^n)$. Consider the ε-regularization $\chi_E * \rho_\varepsilon$ of χ_E, defined as:

$$(\chi_E * \rho_\varepsilon)(x) = \int_{\mathbb{R}^n} \rho_\varepsilon(x-y)\, \chi_E(y)\, dy = \int_{B(x,\varepsilon)} \rho_\varepsilon(x-y)\, \chi_E(y)\, dy$$

$$= \int_{E \cap B(x,\varepsilon)} \rho_\varepsilon(x-y)\, dy$$

$\therefore \quad 0 \le \chi_E * \rho_\varepsilon \le 1$

$$(\chi_E * \rho_\varepsilon)(x) = \begin{cases} 1, & \text{if } |B(x,\varepsilon) \setminus E| = 0 \\ 0, & \text{if } |B(x,\varepsilon) \cap E| = 0 \end{cases}$$



$0 < u_\varepsilon < 1$ on grey band

$B(x,\varepsilon)$

$u_\varepsilon = 0$

If E is an open set with smooth boundary, then:

$$\nabla(\chi_E * \rho_\varepsilon)(x) \approx \frac{1}{\varepsilon}\nu_E(y), \quad y = \text{projection of } x \text{ on } \partial E$$

$$\text{dist}(x, \partial E) < \varepsilon.$$

Hence, as $\varepsilon \to 0$ it should hold that:

$$\int_{\mathbb{R}^n} |\nabla(\chi_E * \rho_\varepsilon)(x)|\,dx \approx \frac{|\{y \in \mathbb{R}^n : \text{dist}(y, \partial E) < \varepsilon\}|}{\varepsilon}$$

$$\approx \frac{\varepsilon \mathcal{H}^{n-1}(\partial E)}{\varepsilon} = \mathcal{H}^{n-1}(\partial E)$$

We now make this rigorous:

__Theorem 1__: Let E be a set of locally finite perimeter in $\mathbb{R}^n$, then

(a) $(\mu_E)_\varepsilon = -\nabla(\chi_E * \rho_\varepsilon)\mathcal{L}^n, \quad \forall \varepsilon > 0$

(b) $(\mu_E)_\varepsilon \xrightarrow{*} \mu_E$

(c) $|\nabla(\chi_E * \rho_\varepsilon)|\mathcal{L}^n \xrightarrow{*} |\mu_E|$

__Proof__: Recall that if μ is a Radon measure on $\mathbb{R}^n$, the the convolution of μ, $\mu_\varepsilon \in C^\infty(\mathbb{R}^n)$, defined as:

$$\mu_\varepsilon(x) = \int_{\mathbb{R}^n} \rho_\varepsilon(x-y)\,d\mu(y),$$

and $\mu_\varepsilon \xrightarrow{*} \mu$ and $|\mu_\varepsilon| \xrightarrow{*} |\mu|$. This applied to μ_E yields $(\mu_E)_\varepsilon \overset{*}{\rightharpoonup} \mu_E$. Also,

$$(\mu_E * \rho_\varepsilon)(x) = \int_{\mathbb{R}^n} \rho_\varepsilon(x-y)\, d\mu_E(y)$$

$$= -\int_E \nabla \rho_\varepsilon(x-y)\, dy \quad ; \text{ since } \int_E \nabla\varphi(x)\,dx = \int_{\mathbb{R}^n} \varphi\, d\mu_E \quad \forall \varphi \in C_c^1(\mathbb{R}^n)$$

$$= -\nabla \int_{\mathbb{R}^n} \rho_\varepsilon(x-y)\chi_E(y)\,dy$$

$$= -\nabla(\chi_E * \rho_\varepsilon)(x) \, ; \quad \text{which is (a)}$$

Since $|(\mu_E)_\varepsilon| \overset{*}{\rightharpoonup} |\mu_E|$ then (c) follows.

Remark 1 : From (c) we have that

$$|(\mu_E)_\varepsilon|(\mathbb{R}^n) \longrightarrow |\mu_E|(\mathbb{R}^n)$$

which is,

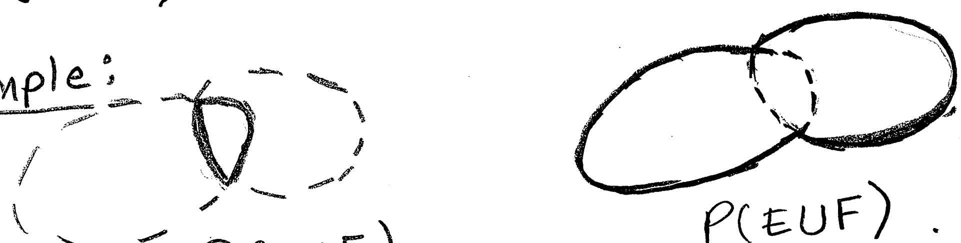
$$\boxed{\int_{\mathbb{R}^n} |\nabla(\chi_E * \rho_\varepsilon)|(x)\,dx \longrightarrow P(E)}$$

With Theorem 1, we can prove a useful result concerning unions and intersections of sets of finite perimeter:

Lemma 1: Let E and F be sets of (locally) finite perimeter in $\mathbb{R}^n$. Then $E \cup F$ and $E \cap F$ are sets of (locally) finite perimeter in $\mathbb{R}^n$, and, for $A \subset \mathbb{R}^n$ open:

$$P(E \cup F; A) + P(E \cap F; A) \leq P(E; A) + P(F; A).$$

Example:



$P(E \cap F)$ $\qquad$ $P(E \cup F)$.

Roughly speaking, if $\mathcal{H}^{n-1}(\partial E \cap \partial F) = 0$ then we actually have the equality:

$$P(E \cup F) + P(E \cap F) = P(E) + P(F)$$

Proof of Lemma 1:

Let $u_\varepsilon = \chi_E * \rho_\varepsilon$, $\quad v_\varepsilon = \chi_F * \rho_\varepsilon$, $\quad 0 \leq u_\varepsilon, v_\varepsilon \leq 1$

$u_\varepsilon v_\varepsilon \longrightarrow \chi_{E \cap F}$ in $L^1_{loc}(\mathbb{R}^n)$

$w_\varepsilon = u_\varepsilon + v_\varepsilon - u_\varepsilon v_\varepsilon \longrightarrow \chi_{E \cup F}$ in $L^1_{loc}(\mathbb{R}^n)$

$$\int_A |\nabla(u_\varepsilon v_\varepsilon)| \leq \int_A v_\varepsilon |\nabla u_\varepsilon| + u_\varepsilon |\nabla v_\varepsilon|$$

$$\int_A |\nabla w_\varepsilon| \leq \int_A (1 - v_\varepsilon)|\nabla u_\varepsilon| + (1 - u_\varepsilon)|\nabla v_\varepsilon|, \quad A \text{ open bounded}$$

$$\Rightarrow \int_A |\nabla(u_\varepsilon v_\varepsilon)| + \int_A |\nabla w_\varepsilon| \leq \int_A |\nabla u_\varepsilon| + |\nabla v_\varepsilon|, \quad A \text{ open bounded}$$

Now;

$$\limsup_{\varepsilon\to 0}\int_A |\nabla u_\varepsilon|+|\nabla v_\varepsilon| \le P(E;\bar{A})+P(F;\bar{A}) \quad (*)$$
$$< \infty$$

; $\Rightarrow$ since $\mu_i \xrightarrow{*} \mu$ implies $\limsup \mu_i(K) \le \mu(K)$, K compact

Now; for every $T\in C_c^1(A;\mathbb{R}^n)$:

$$\int_{\mathbb{R}^n}\chi_{E\cap F}\operatorname{div}T+\int_{\mathbb{R}^n}\chi_{E\cup F}\operatorname{div}T=\lim_{\varepsilon\to\infty}\int_{\mathbb{R}^n}(u_\varepsilon v_\varepsilon+w_\varepsilon)\operatorname{div}T$$

$$=\lim_{\varepsilon\to 0}-\int_{\mathbb{R}^n}T\cdot\nabla(u_\varepsilon v_\varepsilon+w_\varepsilon)$$

$$\le \limsup_{\varepsilon\to 0}\int_A|\nabla u_\varepsilon|+|\nabla v_\varepsilon|<\infty;\ \text{by } (*)$$

Taking the sup over all $T\in C_c^1(A;\mathbb{R}^n)$ we obtain: $\longrightarrow$ see Exercise 12.18

$$P(E\cup F;A)+P(E\cap F;A)\le \limsup_{\varepsilon\to 0}\int_A|\nabla u_\varepsilon|+|\nabla v_\varepsilon|.\ \text{Using } (*):$$

$$(**)\quad P(E\cup F;A)+P(E\cap F;A)\le P(E;\bar{A})+P(F;\bar{A}),\quad \forall A\subset\mathbb{R}^n \text{ open, bounded}$$

Now, let A be any open set in $\mathbb{R}^n$. Let:

$$A_k=\{x\in A\cap B_k : d(x,\partial A)<\tfrac{1}{k}\}\qquad B_k=B(0,k)$$

By $(**)$ applied to each A_k:

$$P(E\cup F;A_k)+P(E\cap F;A_k)\le P(E;\bar{A}_k)+P(F;\bar{A}_k)$$
$$\le P(E;A)+P(F;A).$$

Let $k\to\infty$, since $A_k\subset A_{k+1}$, $A=\bigcup_{k=1}^{\infty}A_k$ and P is a measure we obtain:

$$P(E\cup F;A)+P(E\cap F;A)\le P(E;A)+P(F;A)$$

Theorem 2 (Compactness): If $R>0$ and $\{E_i\}$ are sets of finite perimeter in $\mathbb{R}^n$, with

$$\sup_{i\in\mathbb{N}} P(E_i)<\infty$$

$$E_i\subset B_R,\ \forall i$$

then $\exists E$ of finite perimeter in $\mathbb{R}^n$ and a subsequence $\{E_{i_k}\}$ such that

$$E_{i_k}\to E \text{ in } L^1(\mathbb{R}^n),\quad \mu_{E_{i_k}}\xrightarrow{*}\mu_E,\quad E\subset B_R$$

Proof: We will do the proof in 3 steps:

Step 1:

Claim: If $Q = x+(0,r)^n$ and $u\in C^1(\mathbb{R}^n)$, then

$$\int_{Q(x,r)}|u-(u)_{Q(x,r)}|\,dx\le\sqrt{n}\,r\int_{Q(x,r)}|\nabla u|\,dx,\quad (u)_{Q(x,r)}=\frac{1}{r^n}\int_{Q(x,r)}u\,dx$$

WLOG we can assume $Q(x,r)=(0,1)^n=Q$ and $(u)_Q=0$, and prove that:

$$\int_Q|u|\,dx\le\sqrt{n}\int_Q|\nabla u|\,dx,\quad u\in C^1(\mathbb{R}^n).\quad (A)$$

Indeed, otherwise we define $\tilde{u}(x)=u(rx)$ and $\tilde{v}(x)=\tilde{u}(x)-(\tilde{u})_Q$. Since $(\tilde{v})_Q=0$ we can apply (A) to $\tilde{v}$:

$$\int_Q|\tilde{v}|\le\sqrt{n}\int_Q|\nabla\tilde{v}| \Rightarrow \int_Q|\tilde{u}-(\tilde{u})_Q|\,dx\le\sqrt{n}\int_Q|\nabla\tilde{u}|\,dx$$

$$\therefore \int_{Q(0,1)}\left(u(rx)-\int_{Q(0,1)}u(rx)\,dx\right)dx\le\sqrt{n}\int_Q|\nabla(u(rx))|\,dx=\sqrt{n}\,r\int_{Q(0,1)}|\nabla u(rx)|\,dx$$

Doing change of variables:

$y = rx \quad dy = r^n dx$ yields:

$$\int_{Q(0,r)} \left(u(y) - \frac{1}{r^n} \int_{Q(0,r)} u(y)\,dy \right) dy \le \sqrt{n}\, r \int_{Q(0,r)} |\nabla u(y)|\,dy$$

$$\therefore \int_{Q(0,r)} \left(u(y) - (u)_{Q(0,r)} \right) dy \le \sqrt{n}\, r \int_{Q(0,r)} |\nabla u(y)|\,dy,$$ which is the desired inequality (Note that the same computations work for $Q(x,r)$ instead of $Q(0,r)$).

Thus, we focus on proving that:

$$\int_Q |u|\,dx \le \sqrt{n} \int_Q |\nabla u|\,dx, \quad u \in C^1(\mathbb{R}^n)$$

Finally, since $\sum_{i=1}^n |x_i| \le \sqrt{n} \sqrt{\sum_{i=1}^n x_i^2}$, it suffices to show:

$$\int_Q |u|\,dx \le \sum_{i=1}^n \int_Q \left| \frac{\partial u}{\partial x_i} \right| dx \quad \to (B)$$

We prove (B) by <u>induction</u>:

- If $n = 1$, $(u)_Q = 0 \Rightarrow \exists x_0$ s.t. $u(x_0) = 0$

$$\therefore |u(x) - \underset{\substack{\parallel \\ 0}}{u(x_0)}| \le \int_0^1 |u'|\,dx, \quad x \in Q.$$

$$\therefore \int_0^1 |u(x)|\,dx \le \int_0^1 |u'(x)|\,dx.$$

- For $n \ge 2$, let $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$.

Define: $v(x_1) = \int_{(0,1)^{n-1}} u(x_1, x')\,dx'$.

Thus: (13.8)

$$\int_Q |u|\,dx = \int_{(0,1)}\left(\int_{(0,1)^{n-1}} u(x)\,dx'\right)dx_1$$

$$= \int_{(0,1)}\left(\int_{(0,1)^{n-1}} \left(u(x) - v(x_1) + v(x_1)\right)dx'\right)dx_1$$

$$\leq \int_{(0,1)}\left[\int_{(0,1)^{n-1}} |u(x_1,x') - v(x_1)|\,dx' + |v(x_1)|\right]dx_1$$

$$= \int_{(0,1)}\int_{(0,1)^{n-1}} |u(x_1,x') - v(x_1)|\,dx'dx_1 + \int_{(0,1)} |v(x_1)|\,dx_1$$

$$\leq \int_{(0,1)}\left(\int_{(0,1)^{n-1}} |u(x_1,x') - v(x_1)|\,dx'\right)dx_1 + \int_{(0,1)} |v(x_1)|\,dx_1; \quad \text{since } \underset{n=1}{(v)_{(0,1)}} = \underset{0}{\overset{\|}{(u)_Q}}$$

$$\leq \int_{(0,1)}\left(\sum_{i=2}^{n}\int_{(0,1)^{n-1}} \left|\frac{\partial u}{\partial x_i}\right|dx'\right)dx_1 + \int_{(0,1)} |v'(x_1)|\,dx_1;$$

since: $(u(x_1,x') - v(x_1))_{(0,1)^{n-1}} = 0$ and hypothesis of induction

$$= \sum_{i=2}^{n}\int_{(0,1)}\int_{(0,1)^{n-1}} \left|\frac{\partial u}{\partial x_i}\right|dx'dx_1 + \int_{(0,1)}\left|\int_{(0,1)^{n-1}} \frac{\partial u}{\partial x_1}\,dx'\right|dx_1$$

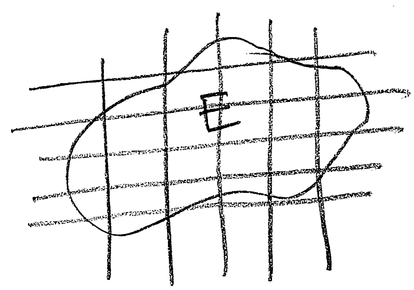
$$\leq \sum_{i=2}^{n}\int_{(0,1)}\int_{(0,1)^{n-1}} \left|\frac{\partial u}{\partial x_i}\right|dx'dx_1 + \int_{(0,1)}\int_{(0,1)^{n-1}} \left|\frac{\partial u}{\partial x_1}\right|dx'dx_1$$

$$= \sum_{i=1}^{n}\int_Q \left|\frac{\partial u}{\partial x_i}\right|dx.$$

<u>Step two</u>: If E is a set of finite perimeter in $\mathbb{R}^n$, $|E| < \infty$, then:

$\forall r > 0$ $\exists T$ union of disjoint cubes of side length r s.t.

$$|E \Delta T| \leq \sqrt{n}\, r\, P(E)$$



Split $\mathbb{R}^n$ as follows:

$\square$ Each cube has side length r.

$$\mathbb{R}^n = \bigcup_{i=1}^{\infty} \overline{Q_i}$$

For $\varepsilon > 0$, let $u_\varepsilon = \chi_E * \rho_\varepsilon$. By step 1:

$$\int_{\mathbb{R}^n} |\nabla u_\varepsilon| = \sum_{i=1}^{\infty} \int_{Q_i} |\nabla u_\varepsilon| \geq \frac{1}{\sqrt{n}\, r} \sum_{i=1}^{\infty} \int_{Q_i} |u_\varepsilon - (u_\varepsilon)_{Q_i}|.$$

From Remark 1: $\displaystyle \int_{\mathbb{R}^n} |\nabla u_\varepsilon| \xrightarrow{\varepsilon \to 0} P(E)$

Also, since $u_\varepsilon \to \chi_E$ in $L^1(\mathbb{R}^n)$: $\displaystyle \sum_{i=1}^{\infty} \int_{Q_i} |u_\varepsilon - (u_\varepsilon)_{Q_i}| \xrightarrow{\varepsilon \to 0} \sum_{i=1}^{\infty} \int_{Q_i} |\chi_E - (\chi_E)_{Q_i}|$

$$\therefore \sqrt{n}\, r \operatorname{Per}(E) \geq \sum_{i=1}^{\infty} \int_{Q_i} |\chi_E - (\chi_E)_{Q_i}|$$

$$= \sum_{i=1}^{\infty} \int_{Q_i} \left| \chi_E - \frac{|Q_i \cap E|}{r^n} \right|$$

$$= \sum_{i=1}^{\infty} \left(\int_{Q_i \cap E} \left| \chi_E - \frac{|Q_i \cap E|}{r^n} \right| + \int_{Q_i \setminus E} \left| \chi_E - \frac{|Q_i \cap E|}{r^n} \right| \right)$$

$$= \sum_{i=1}^{\infty} \int_{Q_i \cap E} \left(1 - \frac{|Q_i \cap E|}{r^n}\right) + \int_{Q_i \setminus E} \frac{|Q_i \cap E|}{r^n}$$

$$= \sum_{i=1}^{\infty} |Q_i \cap E| - \frac{|Q_i \cap E|^2}{r^n} + \frac{|Q_i \cap E|\,|Q_i \setminus E|}{r^n}$$

$$= \sum_{i=1}^{\infty} |Q_i \cap E| \left(1 - \frac{|Q_i \cap E|}{r^n}\right) + \frac{|Q_i \cap E|\,|Q_i \setminus E|}{r^n}$$

$$= \sum_{i=1}^{\infty} \frac{|Q_i \cap E|}{r^n} \left(r^n - |Q_i \cap E| + |Q_i \setminus E| \right)$$

$$= \sum_{i=1}^{\infty} \frac{|Q_i \cap E|}{r^n} \left(|Q_i \setminus E| + |Q_i \setminus E| \right);$$ Since $|Q_i \cap E| + |Q_i \setminus E| = r^n$

$$= 2 \sum_{i=1}^{\infty} \frac{|Q_i \cap E|\,|Q_i \setminus E|}{r^n}$$

$$= 2 \sum_{i=1}^{N} \frac{|Q_i \cap E|\,|Q_i \setminus E|}{r^n} + 2 \sum_{i=N+1}^{\infty} \frac{|Q_i \cap E|\,|Q_i \setminus E|}{r^n};$$

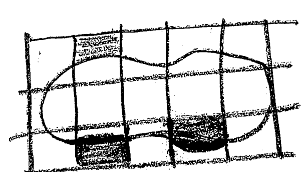
since, up to a permutation, $\exists N$ such that $|Q_i \cap E| \geq \frac{r^n}{2}$ $i = 1, \dots, N$ and $|Q_i \setminus E| \geq \frac{r^n}{2}$, $i = N+1, \dots$ because if N is not finite then $|E|$ would be ∞.

$$\geq \frac{2}{r^n} \sum_{i=1}^{N} \frac{r^n}{2} |Q_i \setminus E| + \frac{2}{r^n} \sum_{i=N+1}^{\infty} \frac{|Q_i \cap E|\, r^n}{2}$$

$$= \sum_{i=1}^{N} |Q_i \setminus E| + \sum_{i=N+1}^{\infty} |Q_i \cap E|$$

$$= |T \setminus E| + |E \setminus T|$$

$$= |T \Delta E|$$



E

$|Q_i \setminus E| \geq r^n/2$ $\quad Q_i \cap E \geq r^n/2$

Let $T := \bigcup_{i=1}^{N} Q_i$

Step three: Define:

$$X = \{E \in \mathcal{M}(\mathcal{L}^n) : |E| < \infty\}$$

X is a complete metric space with distance:

$$d(E,F) = |E \Delta F| = \|\chi_E - \chi_F\|_{L^1(\mathbb{R}^n)}$$

(We identify E and F if $|E \Delta F| = 0$).

Let:

$$Y_{R,p} = \{E \in \mathcal{M}(\mathcal{L}^n) : E \subset B_R,\ P(E) \leq p\},\quad R, p \in (0,\infty).$$

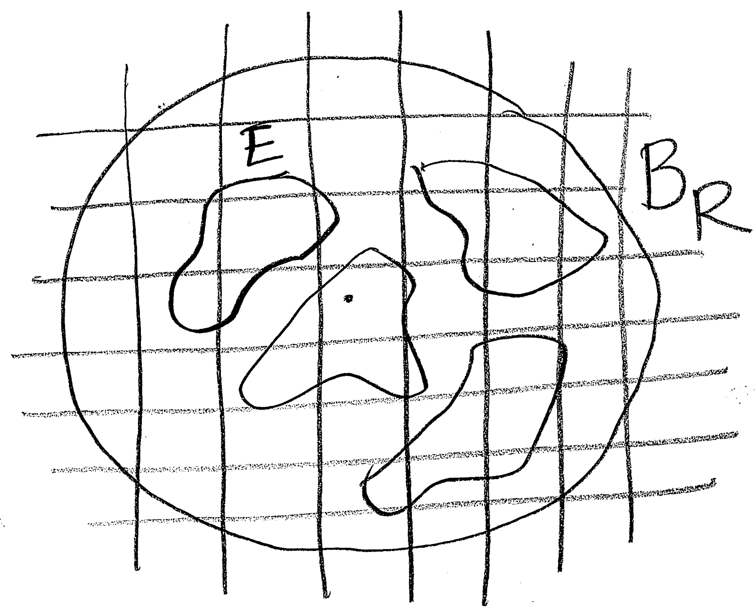
We will show that $Y_{R,p}$ is compact.
By Theorem 3 (Lower semicontinuity of perimeter) proved in Lecture 12, it follows that:

$$\boxed{Y_{R,p} \text{ is closed}} \Rightarrow \boxed{Y_{R,p} \text{ is complete}}$$

Recall:
Thm: (X, ρ) metric space, $A \subset X$. Then:
A is compact $\Leftrightarrow$ A is complete and totally bounded

Def: A is totally bounded if $\forall \varepsilon > 0$, A can be covered by finitely many balls of radius ε.

Thus, given $\varepsilon > 0$, choose r so that $\sqrt{n}\, r p \leq \varepsilon$.
For this r, consider the sequence of cubes $\{Q_i\}_{i=1}^{\infty}$ as in step two, so that the side length of each Q_i is r.



$E \subset \Upsilon_{R,p}$.

$\{Q_i\}$

- Let $\{S_i\}_{i=1}^N$ be the finite family of cubes from $\{Q_i\}_{i=1}^\infty$ that intersect B_R.
- Consider the family $\{T_i\}_{i=1}^M$ of the finite unions of cubes from $\{S_i\}_{i=1}^N$
- Consider now the M balls of radius ε in X:

$$B_\varepsilon(T_i) = \left\{ E \in X : d(E, T_i) \leq \varepsilon \right\}.$$

These M balls cover $\Upsilon_{R,p}$ because, given any $E \in \Upsilon_{R,p}$, $E \subset B_R$, and by Step 2, $\exists\, T_i$, $i \in \{1, \dots, M\}$ s.t. that:

$$d(E, T_i) = |E \Delta T_i| \leq \sqrt{n}\, r P(E) \leq \sqrt{n}\, r p \leq \varepsilon$$

$$\Rightarrow \quad E \in B_\varepsilon(T_i).$$

We conclude that $\Upsilon_{R,p}$ is compact in X.

Step four: We finally prove our compactness theorem:

We have:

$$E_i \subset B_R, \quad P(E_i) \le p, \quad \text{some } p, \ \forall i$$

$$\Rightarrow \quad E_i \in \Upsilon_{R,p}$$

Since $\Upsilon_{R,p}$ is compact, $\exists \{E_{i_k}\}$ and $E \in \Upsilon_{R,p}$ such that:

$$d(E_{i_k}, E) \xrightarrow{k \to \infty} 0$$

$$\therefore \quad \boxed{E_{i_k} \to E \text{ in } L^1(\mathbb{R}^n)}$$

By Theorem 3 (The lower semicontinuity of the perimeter) proved in Lecture 12 it follows that:

$$P(E) \le \liminf_{k \to \infty} P(E_{i_k}) \le p$$; we had seen before that $\Upsilon_{R,p}$ is closed

$$\boxed{E \text{ is a set of finite perimeter in } \mathbb{R}^n}$$

and:

$$\boxed{\mu_{E_{i_k}} \xrightarrow{*} \mu_E}$$

∎