Lecture 15

In this lecture we will prove the coarea formula:

Theorem (Coarea formula): If $u: \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz function and $A \subset \mathbb{R}^n$ is open, then $t \mapsto P(\{u>t\}; A)$ is a Borel function on $\mathbb{R}$ with:

$$\int_A |\nabla u|\, dx = \int_{\mathbb{R}} P(\{u>t\}; A)\, dt$$

as elements of $[0,\infty]$.

Proof:

Step 1: Claim: $t \mapsto P(\{u>t\}; A)$ is a Borel function.

Let $T \in C_c^1(A; \mathbb{R}^n)$. Note that we can write

$$\int_{\{u>t\}} \operatorname{div} T = \int_{\{u>t\}} (\operatorname{div} T)^+ - \int_{\{u>t\}} (\operatorname{div} T)^- ;$$

where $t \mapsto \int_{\{u>t\}} (\operatorname{div} T)^+$ and $t \mapsto \int_{\{u>t\}} (\operatorname{div} T)^-$ are non-increasing functions.

Refer to Chapter 7 of "Modern Real Analysis", which you can find on my website, for the proof of the following: A bounded function that is either non-increasing or non-decreasing is of bounded variation. Moreover, a function f of bounded variation on $[a,b]$ is Borel measurable and $f'(x)$ exists almost everywhere on $[a,b]$. Using this it follows that:

$$t \mapsto \int_{\{u>t\}} \operatorname{div} T \quad \text{is Borel measurable.}$$

(5.2)

If $\mathcal{F}$ is a countable dense subset of $C_c^\infty(A;\mathbb{R}^n)$, then:

$$P(\{u>t\};A) = \sup\left\{ \int_{\{u>t\}} \operatorname{div} T : T \in \mathcal{F} : |T| \le 1 \right\}$$

Since the sup of measurable functions (countable meas. functions) is measurable we proved the claim; that is,

$$t \mapsto P(\{u>t\};A) \text{ is Borel measurable.}$$

(see Section 5.2 of "Modern Real Analysis" for the theory of measurable functions).

Step two: If $u \geq 0$ Lipschitz then

$$\int_A |\nabla u| \le \int_0^\infty P(\{u>t\};A)\,dt, \quad \forall A \subset \mathbb{R}^n \text{ open}$$

In particular, if the left-hand side is infinite then the right hand side is infinite too.

Before proving step two we recall the following properties of Lipschitz functions. If $f: \mathbb{R}^n \to \mathbb{R}$ is Lipschitz then the distributional gradient, ∇f, belongs to $L^\infty(\mathbb{R}^n; \mathbb{R}^n)$; which means:

$f: \mathbb{R}^n \to \mathbb{R}$ lip $\Rightarrow$ $f \in L^\infty_{loc}(\mathbb{R}^n)$, $\nabla f \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$

$$(*) \qquad \int_{\mathbb{R}^n} f \, \nabla \varphi = - \int_{\mathbb{R}^n} \varphi \, \nabla f, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^n),$$

or, equivalently:

$$(**) \qquad \int_{\mathbb{R}^n} f \operatorname{div} T = - \int_{\mathbb{R}^n} T \cdot \nabla f, \quad \forall T \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$$

Moreover, if x is a Lebesgue point of ∇f, then f is differentiable at x. In particular, f is differentiable a.e. on $\mathbb{R}^n$ (Rademacher's theorem).

We now fix $T \in C_c^\infty(A; \mathbb{R}^n)$, $|T| \leq 1$. Then, by definition of perimeter:

$$\int_{\{u>t\}} \operatorname{div} T \leq P(\{u>t\}; A), \quad t>0.$$

By $(**)$:

$$-\int_A \nabla u \cdot T = \int_{\mathbb{R}^n} u \operatorname{div} T = \int_0^\infty \left(\int_{\{u>t\}} \operatorname{div} T \right) dt \leq \int_0^\infty P(\{u>t\}; A) \, dt, \quad (1)$$

See Lemma 1 below

where we have used the following:

<u>Lemma 1</u>: If $u \in L^1(\mathbb{R}^n)$, $u \geq 0$, $v \in L^\infty(\mathbb{R}^n)$,

then $$\int_{\mathbb{R}^n} u(x)v(x) = \int_0^\infty \left(\int_{\{u>t\}} v(x)\,dx \right) dt$$

Indeed, for every $x \in \mathbb{R}^n$,

$$u(x) = \int_{\mathbb{R}} \chi_{(0,u(x))}\,dt = \int_{\mathbb{R}} \chi_{(0,\infty)}(t)\,\chi_{\{u>t\}}(x)\,dt$$
$$= \int_0^\infty \chi_{\{u>t\}}(x)\,dt,$$

and thus, by Fubini's theorem:

$$\int_{\{u \geq 0\}} u(x)v(x) = \int_{\{u\geq 0\}} v(x) \left(\int_0^\infty \chi_{\{u>t\}}(x)\,dt \right) dx$$
$$\overset{\text{Fubini}}{=} \int_0^\infty \left(\int_{\{u>t\}} v(x)\,dx \right) dt$$

We now substitute in (1) a particular T defined as follows: Let K be a compact subset of A and define $S : \mathbb{R}^n \to \mathbb{R}^n$ as:

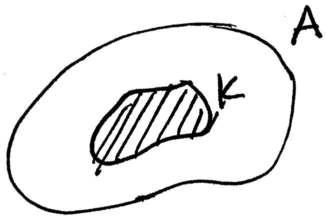
$$S(x) = -\chi_{K \cap \{\nabla u \neq 0\}}(x) \, \frac{\nabla u(x)}{|\nabla u(x)|}, \quad x \in \mathbb{R}^n$$

S is a bounded Borel measurable vector field with $|S| \leq 1$. For ε small enough we have:

$$S_\varepsilon = S * \rho_\varepsilon \in C_c^\infty(A;\mathbb{R}^n),$$

and

$|S_\varepsilon| \le 1$, $S_\varepsilon(x) \to S(x)$ a.e. $x \in \mathbb{R}^n$.



For $\varepsilon < \operatorname{dist}(K, \partial A)$, the convolution has support inside A.

We let $T_\varepsilon := S_\varepsilon$ in (1):

$$-\int_A \nabla u \cdot T_\varepsilon \le \int_0^\infty P(\{u>t\}; A)\, dt,$$

Letting $\varepsilon \to 0$

$$-\int_A \nabla u \cdot S \le \int_0^\infty P(\{u>t\}; A)\, dt \; ; \quad T_\varepsilon \to S \text{ a.e. } x, \; |T_\varepsilon| \le 1$$

$\Rightarrow$

$$\int_K \nabla u \cdot \frac{\nabla u}{|\nabla u|} \le \int_0^\infty P(\{u>t\}; A)\, dt$$

$$\| $$

$$\int_K |\nabla u| ,$$

where $\int_K |\nabla u|\, dx < \infty$ since $\nabla u \in L^\infty(\mathbb{R}^n; \mathbb{R}^n)$.

Since K is arbitrary we conclude:

$$\int_A |\nabla u| \le \int_0^\infty P(\{u>t\}; A)\, dt .$$

Step three: We prove that if $u \geq 0$ is Lipschitz then:

$$\int_A |\nabla u| \geq \int_0^\infty P(\{u>t\}; A)\, dt$$

Consider the increasing function $m: \mathbb{R} \to [0,\infty)$ as:

$$m(t) = \int_{A \cap \{u \leq t\}} |\nabla u| \qquad t \in \mathbb{R}.$$

As mentioned before in this lecture, since m is increasing $\Rightarrow$ m is differentiable for a.e. t (see Section 7.3 of "Modern Real Analysis for a proof"). Moreover:

$$\underset{\displaystyle\int_0^\infty m'(t)\,dt}{\underset{\|}{\int_{\mathbb{R}} m'(t)\,dt}} \leq \underset{\displaystyle = \int_A |\nabla u|}{\lim_{t\to\infty} m(t)} - \overset{0}{\overset{\|}{\lim_{t\to-\infty} m(t)}} ;$$

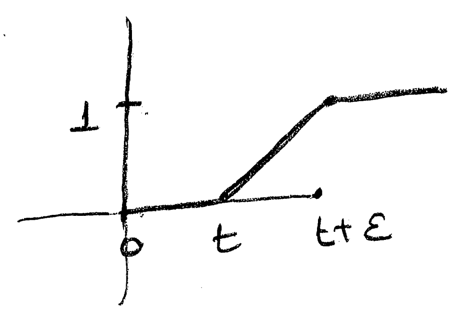
see Exercise 5.14 in textbook.

Thus, it is enough to prove that:

$$m'(t) \geq P(\{u>t\}; A), \text{ for a.e. } t \geq 0.$$

Given $t \geq 0$, $\varepsilon \geq 0$, define $\psi: [0,\infty) \to [0,1]$ as:

$$\psi_\varepsilon(s) = \begin{cases} 1, & t+\varepsilon \leq s < \infty \\ \dfrac{s-t}{\varepsilon}, & t \leq s < t+\varepsilon \\ 0, & 0 \leq s < t \end{cases}$$



We now need the following chain rule for weak gradients: We say that $f:\mathbb{R}\to\mathbb{R}$ is a piecewise affine function if f is continuous and if there is a partition of $\mathbb{R}$ into finitely many intervals such that f is affine on each interval of the partition. Note that, if f is piecewise affine, then there exists a finite set F such that $f'(s)$ exists for every $s\in\mathbb{R}\setminus F$. Moreover, if $u\in W^{1,1}_{loc}(\mathbb{R}^n)$, then $\nabla u=0$ a.e. on $u^{-1}(F)$ and $(f\circ u)\in W^{1,1}_{loc}(\mathbb{R}^n)$ with:

$$\boxed{\nabla(f\circ u)=(f'\circ u)\nabla u} \quad (2)$$

Going back to the proof of Step 3, we apply (2) to $\gamma_\varepsilon\circ u$ to get

$$\nabla(\gamma_\varepsilon\circ u)(x)=-\frac{1}{\varepsilon}\underbrace{\chi_{(t,t+\varepsilon)}(u(x))}_{=1 \text{ if } t<u(x)<t+\varepsilon}\nabla u(x) \quad \text{a.e. } x.$$

If $T\in C^\infty_c(A;\mathbb{R}^n)$, $|T|\le 1$ $\Rightarrow$

$$\int_A(\gamma_\varepsilon\circ u)\operatorname{div}T=-\frac{1}{\varepsilon}\int_{A\cap\{t<u<t+\varepsilon\}}\nabla u\cdot T \quad ; \text{ by } (**)$$

$$\le\frac{1}{\varepsilon}\int_{A\cap\{t<u<t+\varepsilon\}}|\nabla u|$$

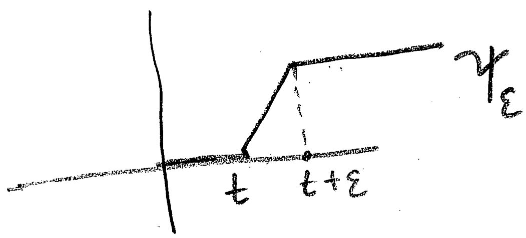
$$\le\frac{1}{\varepsilon}\left(m(t+\varepsilon)-m(t)\right); \quad \text{since: } m(t)=\int_{A\cap\{u\le t\}}|\nabla u|$$

We have:

$$\int_A (\psi_\varepsilon \circ u)\, \operatorname{div} T \le \frac{1}{\varepsilon}\left(m(t+\varepsilon) - m(t)\right) \quad (3)$$

Since $(\psi_\varepsilon \circ u)(x) \to \chi_{A \cap \{u>t\}}(x)$, a.e. x.

Indeed, $x \in A \Rightarrow u(x) > t$ or $u(x) \le t$, if $u(x) \le t \Rightarrow \psi_\varepsilon(u(x)) = 0 \to \chi_{A\cap\{u>t\}}(x) = 0$. If $u(x) > t$ then, for ε small enough, $\psi_\varepsilon(u(x)) = 1 \to \chi_{A\cap\{u>t\}}(x) = 1$.



Letting $\varepsilon \to 0$ in (3) yields, by the dominated convergence theorem:

$$\int_{A\cap\{u>t\}} \operatorname{div} T \le m'(t), \quad \text{a.e. } t > 0$$

Taking the sup over all $T \in C_c^1(A; \mathbb{R}^n)$, $|T| \le 1 \Rightarrow$

$$P(\{u>t\}; A) \le m'(t), \quad \text{a.e. } t. \quad (4)$$

Putting all together:

$$\int_A |\nabla u| = \lim_{t\to\infty} m(t) \ge \int_0^\infty m'(t)\, dt \ge \int_0^\infty P(\{u>t\}; A)\, dt \text{ ; by (4).}$$

That is: $\displaystyle\int_A |\nabla u| \ge \int_0^\infty P(\{u>t\}; A)\, dt$. ∎

Step four : Given, in general, $u : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function, we write:

$$u = u^+ - u^-, \quad u^+ = \max\{u, 0\} \in W^{1,1}_{loc}(\mathbb{R}^n)$$
$$u^- = \max\{-u, 0\} \in W^{1,1}_{loc}(\mathbb{R}^n)$$

It is proved in Section 7.2 of our textbook (Maggi's book) that:

$$\nabla u^+ = \chi_{\{u>0\}} \nabla u, \quad \nabla u^- = -\chi_{\{u<0\}} \nabla u$$

By Step two and three, since $u^+, u^- \geq 0$:

$$\int_A |\nabla u^+| = \int_0^\infty P(\{u^+ > t\}; A)\, dt, \quad \int_A |\nabla u^-| = \int_0^\infty P(\{u^- > t\}; A)\, dt$$

and thus:

$$\int_A |\nabla u| = \int_A |\nabla u^+| + \int_A |\nabla u^-|$$
$$= \int_0^\infty P(\{u^+ > t\}; A)\, dt + \int_0^\infty P(\{u^- > t\}; A]\, dt$$
$$= \int_0^\infty P(\{u > t\}; A)\, dt + \int_0^\infty P(\{u < -t\}; A)\, dt$$
$$= \int_0^\infty P(\{u > t\}; A)\, dt + \int_{-\infty}^0 P(\{u < s\}; A)\, ds, \quad -t = s$$
$$= \int_0^\infty P(\{u > t\}; A)\, dt + \int_{-\infty}^0 P(\{u < t\}; A)\, dt \quad \to (5)$$

As we have mentioned before, note from the definition of perimeter that

$$\mu_{\mathbb{R}^n \setminus E} = -\mu_E \quad \Big(\int_E \operatorname{div} T = \underbrace{\int_{\mathbb{R}^n} \operatorname{div} T}_{=0} + \int_{\mathbb{R}^n \setminus E} \operatorname{div} T, \ \forall T \in C^1_c(\mathbb{R}^n) \Big).$$

and hence:

$$P(E) = P(\mathbb{R}^n \setminus E) \quad \text{and also}$$

$$P(E; A) = P(\mathbb{R}^n \setminus E; A), \quad \forall A \subset \mathbb{R}^n \text{ open}.$$

Thus:

$$P(\{u < t\}; A) = P(\{u \geq t\}; A)$$

Now, by foliations by Radon measures (see Lecture 2, Page 2.7) we have:

$$\mathcal{L}^n(\{u = t\}) = 0 \quad \text{for a.e. } t \in \mathbb{R}$$

and hence:

$$P(\{u \geq t\}; A) = P(\{u > t\}; A) \text{ for a.e. } t \in \mathbb{R}.$$

Therefore, from (5):

$$\int_A |\nabla u| = \int_0^\infty P(\{u > t\}; A)\, dt + \int_{-\infty}^0 P(\{u > t\}; A)\, dt$$

$$= \int_{-\infty}^\infty P(\{u > t\}; A)\, dt$$

We have completed the proof of the Coarea formula. ∎

Remark 1: A useful application of Coarea is:

$$|E \cap B(x,r)| = \int_0^r \mathcal{H}^{n-1}(E \cap \partial B(x,t))\, dt, \quad \forall E \text{ Borel}$$

Proof: Use Coarea with Lipschitz function $u(y) = |y - x|$, $y \in \mathbb{R}^n$, $|\nabla u(y)| = 1$.

Remark 2: If $u \in C^\infty(\mathbb{R}^n)$, we have the coarea formula $\forall E$ Borel set. That is, $\int_E |\nabla u| = \int_{\mathbb{R}} \mathcal{H}^{n-1}(E \cap \{u = t\})\, dt$.

Approximation of sets of finite perimeter: When working with sets of finite perimeter, one often needs to approximate them with smooth sets. Actually, one can find different types of approximations in the literature (by polyhedra, from the "interior", from the "exterior", etc), but the following is a fundamental approximation theorem:

Thm (Approximation by smooth sets): A Lebesgue measurable set $E \subset \mathbb{R}^n$ is of locally finite perimeter if and only if there exists a sequence $\{E_k\}_{k=1}^{\infty}$ of open sets with smooth boundary in $\mathbb{R}^n$, and $\varepsilon_k \to 0$, such that:

- $E_k \to E$ in $L^1_{loc}(\mathbb{R}^n)$, $\sup P(E_k, B_R) < \infty$ $\forall R > 0$

- $|\mu_{E_k}| \overset{*}{\rightharpoonup} |\mu_E|$, $\partial E_k \subset I_{\varepsilon_k}(\partial E)$

In particular:

- $P(E_k; F) \to P(E; F)$, if $P(E; \partial F) = 0$
- If $|E| < \infty$ then $E_k \to E$ in $L^1(\mathbb{R}^n)$
- If $P(E) < \infty$ then $P(E_k) \to P(E)$.

Sketch of proof:

Given $\varepsilon > 0$, $\varepsilon_i \to 0$, $t \in (0,1)$

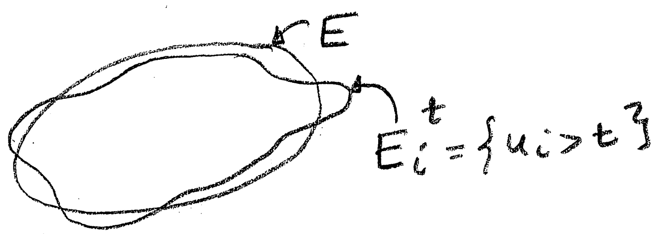
we define:

$$u_\varepsilon := \chi_E * \rho_\varepsilon, \quad u_i := u_{\varepsilon_i}, \quad E_i^t = \{u_i > t\}.$$

Then:

$$E_i^t \longrightarrow E \quad \text{in } L^1_{loc}(\mathbb{R}^n), \quad \text{a.e. } t \in (0,1)$$

$$P(E;A) \leq \liminf_{i \to \infty} P(E_i^t;A)$$



We now show that for a.e. $t \in (0,1)$:

$$P(E;A) = \liminf_{i \to \infty} P(E_i^t;A)$$

We have:

$$P(E;A) \leq \int_0^1 \liminf_{i \to \infty} P(E_i^t;A)$$

On the other hand,

$$P(E;A) = \lim_{i \to \infty} \int_A |\nabla u_i| \;; \text{ by Remark 1, Lecture 13.}$$

$$= \lim_{i \to \infty} \int_{\mathbb{R}} P(E_i^t;A)\,dt; \text{ by Coarea formula}$$

$$= \lim_{i \to \infty} \int_0^1 P(E_i^t;A)\,dt, \text{ since } 0 \leq u_i \leq 1$$

$$P(E;A) = \lim_{i\to\infty} \int_0^1 P(E_i^t; A)\,dt$$

$$= \liminf_{i\to\infty} \int_0^1 P(E_i^t; A)\,dt$$

$$\geq \int_0^1 \liminf_{i\to\infty} P(E_i^t; A)\,dt \quad ; \text{ Fatou's Lemma.}$$

Therefore:

$$\boxed{P(E;A) = \int_0^1 \liminf_{i\to\infty} P(E_i^t; A)}$$

Hence:

$$\int_0^1 \Big(\underbrace{\liminf_{i\to\infty} P(E_i^t; A) - P(E;A)}_{\geq 0}\Big)\,dt = 0$$

$$\therefore \boxed{P(E;A) = \liminf_{i\to\infty} P(E_i^t; A) \quad \text{for a.e. } t} \qquad (6)$$

Thus, if $|E| < \infty$ then $u_i \to \chi_E$ in $L^1(\mathbb{R}^n)$ and $E_i^t \to E$ for a.e. $t \in (0,1)$. If $P(E) < \infty$, then with $A = \mathbb{R}^n$ in (6), there exists a subsequence $\{E_{i_k}^t\}$ (we can pick a t, since (6) is true for a.e. t):

$$P(E_k) \to P(E) \quad ; \quad E_k := E_{i_k}^t,$$

In the case when E is of <u>locally</u> finite perimeter, we apply (6) on balls $\{B_{r_i}\}$, $r_i \to \infty$ and a diagonal argument to get $\{E_k\}$ s.t. $P(E; B_{r_i}) = \lim_{k\to\infty} P(E_k; B_{r_i})$ $\forall i$. ∎