Lecture 17

In this lecture we will prove Steiner inequality (Theorem 2, Lecture 16), which was the main ingredient in the proof of the isoperimetric inequality.

<u>Step one</u>: In this step we show that

$$P(E^s) \leq P(E)$$

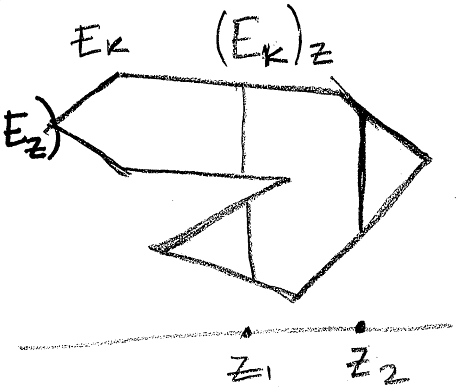
Since $P(E)<\infty$, $|E|<\infty$, by Corollary 2 in Lecture 16, there exists $\{E_k\}_{k=1}^{\infty}$ bounded open sets with polyhedral boundary such that, as $k\to\infty$,

$$\boxed{E_k \to E \text{ in } L^1(\mathbb{R}^n), \quad P(E_k)\to P(E)} \quad (1)$$

Let:

$$m_k(z)=\mathcal{L}^1((E_k)_z), \quad m(z)=\mathcal{L}^1(E_z)$$

$$G_k=\{z\in\mathbb{R}^{n-1}: m_k(z)>0\}$$



$$D_k=\{z\in\mathbb{R}^{n-1}: (E_k)_z\subset\mathbb{R} \text{ is not an interval}\}$$

$z_1\in D_k$
$z_2\notin D_k$

$$G=\{z\in\mathbb{R}^{n-1}: m(z)>0\}.$$

Note that ν_{E_k} takes only finitely many values, and hence, up to rotating each E_k by a rotation sufficiently close to the identity, we can assume

$$\boxed{\nu_{E_k}\cdot e_n\neq 0} \quad (2)$$

We will prove below that a bounded set with polyhedral boundary satisfying (2) has the following properties: (17.2)

$$\boxed{\begin{aligned} &P(E_k^s) \le P(E_k) \\ &2\mathcal{H}^{n-1}(D_k)^2 \le P(E_k)\left(P(E_k) - P(E_k^s)\right) \end{aligned}} \quad (3)$$

By Fubini's Theorem:

$$|E_k \Delta E| = \int_{\mathbb{R}^{n-1}} \mathcal{L}^1((E_k)_z \Delta E_z)\,dz \ge \int_{\mathbb{R}^{n-1}} |m_k(z) - m(z)|\,dz = |E_k^s \Delta E^s|$$

Thus, $|E_k \Delta E| \to 0$ yields $E_k^s \to E^s$ in $L^1(\mathbb{R}^n)$

$$\Rightarrow \boxed{P(E^s) \le \liminf_{k\to\infty} P(E_k^s)} \quad (4)$$

Since $P(E_k) \to P(E)$; from (3) and (4):

$$2 \limsup_{k\to\infty} \mathcal{H}^{n-1}(D_k)^2 \le P(E)\left(\limsup_{k\to\infty} P(E_k) + \limsup_{k\to\infty}(-P(E_k^s))\right)$$

$$= P(E)\left(P(E) - \liminf_{k\to\infty} P(E_k^s)\right)$$

$$\le P(E)\left(P(E) - P(E^s)\right)$$

Thus:

$$\boxed{2 \limsup_{k\to\infty} \mathcal{H}^{n-1}(D_k)^2 \le P(E)\left(P(E) - P(E^s)\right)} \quad (5)$$

Clearly, from (5):

$$\boxed{P(E^s) \leq P(E)}$$

, which is (***) in Theorem 2.

We now prove (i) in Theorem 2. Indeed, if $P(E) = P(E^s)$ then:

$$\limsup_{K\to\infty} \mathcal{H}^{n-1}(D_K)^2 = 0;$$

that is,

$$\lim_{K\to\infty} \mathcal{H}^{n-1}(D_K) = 0 \Rightarrow \boxed{\chi_{D_K} \to 0 \text{ in } L^1(\mathbb{R}^{n-1})} \quad (6)$$

Now:

$\int_{\mathbb{R}^{n-1}} \mathcal{L}^1((E_K)_z \Delta E_z)\, dz = 0 \to 0$ as $K \to \infty$ implies that there exists a subsequence of $\{E_K\}$, denoted again as $\{E_K\}$, such that:

$$\mathcal{L}^1((E_K)_z \Delta E_z) \to 0 \text{ for a.e. } z \in \mathbb{R}^{n-1}.$$

$$\boxed{\therefore\ \chi_{(E_K)_z} \to \chi_{E_z} \text{ in } L^1(\mathbb{R}), \text{ for a.e. } z \in \mathbb{R}^{n-1}} \quad (7)$$

And also:

$$\boxed{\chi_{G_K} \to \chi_G \text{ in } L^1(\mathbb{R}^{n-1})} \quad (8)$$

Now;

$$\chi_{(E_K)_z} \to \chi_{E_z} \text{ in } L^1(\mathbb{R}) \quad \text{implies}$$

$$P(E_z) \leq \liminf_{K\to\infty} P((E_K)_z) \text{ ; and this is true for a.e. } z$$

From (6) and (8):

$$\chi_{G_K \setminus D_K} \to \chi_G \text{ in } L^1(\mathbb{R}^{n-1}) \text{ ; }$$

recall that $\mathcal{H}^{n-1}(D_K) \to 0$ and thus $\mathcal{H}^{n-1}(G_K \cap (\mathbb{R}^n \setminus D_K))$

$$= \underbrace{\mathcal{H}^{n-1}(G_K)}_{\downarrow \; \mathcal{H}^{n-1}(G)} - \underbrace{\mathcal{H}^{n-1}(G_K \cap D_K)}_{\downarrow \; 0}.$$

as $K\to\infty$

Note that $\mathcal{H}^{n-1}(D_K) \to 0$ means that as $K \to \infty$, "most" of the sections $(E_K)_z$ are intervals.

From $\chi_{G_K \setminus D_K} \to \chi_G$ in $L^1(\mathbb{R}^{n-1})$, we have that, for a further subsequence:

$$\lim_{K\to\infty} \chi_{G_K \setminus D_K}(z) = \chi_G(z) \text{ , } \mathcal{H}^{n-1}\text{-a.e. } z.$$

Thus; multiplying by $\chi_G(z)$ in above inequality:

$$\chi_{G}(z) P(E_z) \leq \chi_G(z) \liminf_{K\to\infty} P((E_K)_z)$$

$$= \left(\lim_{K\to\infty} \chi_{G_K \setminus D_K}(z)\right)\left(\liminf_{K\to\infty} P((E_K)_z)\right); \text{ a.e. } z$$

$$\leq \liminf_{K\to\infty}\left(\chi_{G_K \setminus D_K}(z)\, P((E_K)_z)\right); \text{ a.e } z$$

where we have used:

$$\liminf_{K\to\infty} (a_K b_K) \geq \liminf_{K\to\infty} a_K \liminf_{K\to\infty} b_K .$$

We have:

$$\chi_G(z)\, P(E_z) \leq \liminf_{K\to\infty} \chi_{G_K \setminus D_K}(z)\, P((E_K)_z) \;, \quad a.e.\ z$$

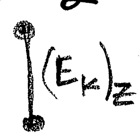
$\Rightarrow$

$$\int_G P(E_z)\, dz \leq \int_{G_K\setminus D_K} \liminf_{K\to\infty} P((E_K)_z)\, dz$$

$$\leq \liminf_{K\to\infty} \int_{G_K\setminus D_K} P((E_K)_z)\, dz \; ; \text{ Using Fatou's lemma}$$

$$= \liminf_{K\to\infty} \int_{G_K\setminus D_K} 2\, dz \; ;$$

Since $(E_K)_z$ is an interval for $z \in D_K$, and hence it has perimeter 2 in $\mathbb{R}$.



$$= 2 \liminf_{K\to\infty} \mathcal{H}^{n-1}(G_K \setminus D_K)$$

$$= 2\mathcal{H}^{n-1}(G).$$

$$\therefore \boxed{\int_G P(E_z)\, dz \leq 2\mathcal{H}^{n-1}(G)} \quad (9)$$

We are going to use now the following Proposition (see textbook):

<u>Proposition 1 (Sets of finite perimeter in $\mathbb{R}$)</u>: $E \subset \mathbb{R}$ is of locally finite perimeter if and only if it is equivalent to a countable union of (possibly unbounded) open intervals lying at mutually positive distance.

Clearly, if $E \subset \mathbb{R}$, $\mathcal{L}^1(E) < \infty$ then
$P(E) \geq 2$

Thus; going back to (9):
$P(E_z) - 2 \geq 0$, and hence:

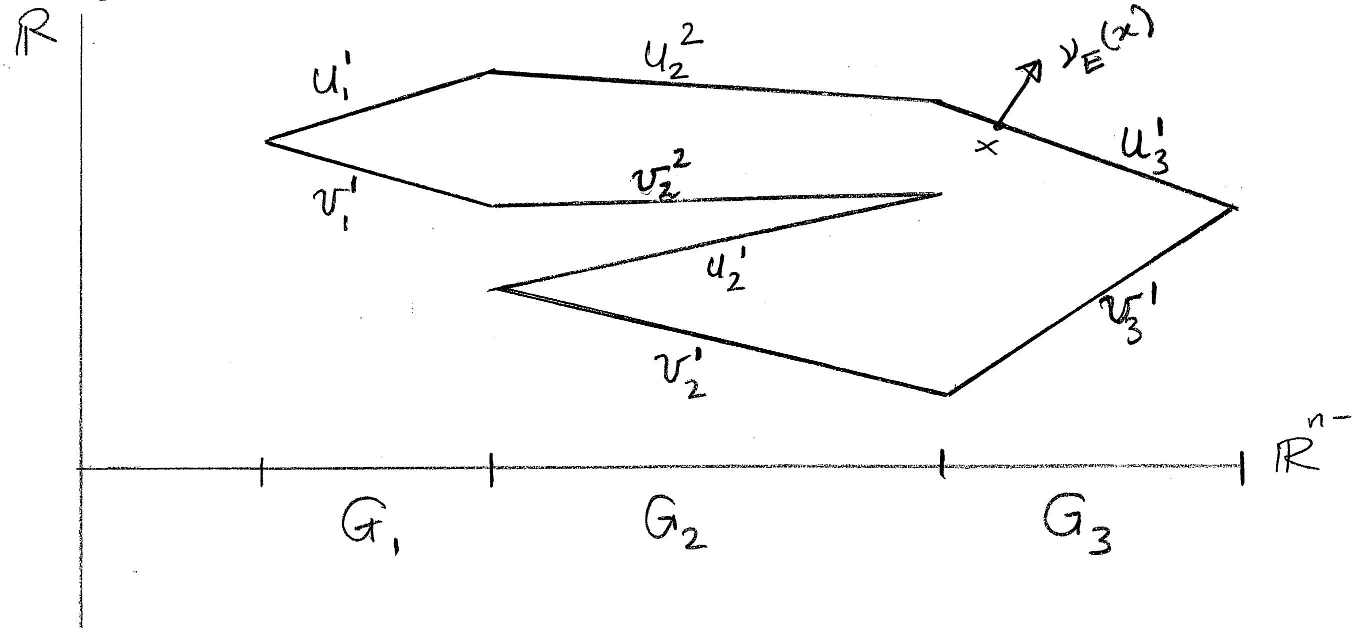
$$\int_G (P(E_z) - 2)\, d\mathcal{H}^{n-1}(z) = 0$$

implies

$$P(E_z) = 2 \quad \text{for a.e. } z \in G.$$

By proposition 1 we have that E_z is equivalent to a countable union of open intervals. But, since $P(E_z) = 2$, we conclude that such union consists of only one interval. Hence:
E_z is equivalent to an open interval, a.e. z.

We have proved (i) in Theorem 2, but we are left to prove that (3) holds for any bounded set with polyhedral boundary:



We assume that $\nu_E(x) \cdot e_n \neq 0$, $\forall x \in \partial E$, $\nu_E(x)$ exterior unit normal

We have:

$$G = \bigcup_{i=1}^{M} G_i$$

and affine functions $v_i^k, u_i^k : G_i \to \mathbb{R}$, $1 \le i \le M$, $1 \le k \le N(i)$, with

$$\partial E = \bigcup_{i=1}^{M} \bigcup_{k=1}^{N(i)} \Gamma(u_i^k, G_i) \cup \Gamma(v_i^k, G_i),$$

$$E = \bigcup_{i=1}^{M} \left\{ (z,t) \in G_i \times \mathbb{R} : t \in \bigcup_{k=1}^{N(i)} (v_i^k(z), u_i^k(z)) \right\}$$

Note:

- $m(z) = \sum_{k=1}^{N(i)} u_i^k(z) - v_i^k(z), \ \forall z \in G_i.$
- m is continuous, piecewise affine

Note: This partition exists because of the assumption $\nu_E(x) \cdot e_n \neq 0$, $\forall x \in \partial E$ and the implicit function theorem.

We will use the following theorem (see Chapter 9 in textbook):

Thm (Area of a graph of codimension one). If $u : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function, then for every Lebesgue measurable set G in $\mathbb{R}^{n-1}$,

$$\mathcal{H}^{n-1}(\Gamma(u;G)) = \int_G \sqrt{1 + |\nabla' u(z)|^2}\, dz$$

(To prove this theorem, apply the area formula to the Lipschitz function $f(z) = (z, u(z))$, $z \in \mathbb{R}^{n-1}$ and compute

the Jacobian of f as (17.8)
$Jf = \sqrt{(\nabla f)^* (\nabla f)}$, which is $Jf = \sqrt{1 + |\nabla' u|^2}$).

Note that:

$$E^s = \{ (z,t) \in G \times \mathbb{R},\ |t| < \frac{m(z)}{2} \}$$

E^s is a bounded open set with pohyhedral boundary.

Using the above formula to compute the area of a graph we have:

$$P(E^s) = \mathcal{H}^{n-1}(\partial E^s) = 2\int_G \sqrt{1 + |\frac{\nabla m}{2}|^2} = \sum_{i=1}^{M} \sqrt{4 + |\nabla m|^2}.$$

$$P(E) = \sum_{i=1}^{M} \int_{G_i} \sum_{K=1}^{N(i)} \sqrt{1 + |\nabla u_i^K|^2} + \sqrt{1 + |\nabla v_i^K|^2}\ dz$$

Since $z \longmapsto \sqrt{1 + |z|^2}$ is convex we have:

$$\sum_{K=1}^{N(i)} \sqrt{1 + |\nabla u_i^K|^2} + \sqrt{1 + |\nabla v_i^K|^2} \geq 2 \sum_{K=1}^{N(i)} \sqrt{1 + \left| \frac{\nabla u_i^K - \nabla v_i^K}{2} \right|^2}$$

Recall; f convex $\Rightarrow f((1-\lambda)x + \lambda y) \leq (1-\lambda) f(x) + \lambda f(y)$, $\lambda = \frac{1}{2} \Rightarrow f(\frac{1}{2}x + \frac{1}{2}y) \leq \frac{1}{2} f(x) + \frac{1}{2} f(y)$

$$= 2N(i) \left\{ \frac{1}{N(i)} \sum_{K=1}^{N(i)} \sqrt{1 + \left| \frac{\nabla u_i^K - \nabla v_i^K}{2} \right|^2} \right.$$

f convex, $\lambda_1 + \lambda_2 + \cdot + \lambda_p = 1 \Rightarrow f(\lambda_1 x_1 + \cdot + \lambda_p x_p) \leq \lambda_1 f(x_1) + \dots + \lambda_p f(x_p)$

$$\geq 2N(i) \sqrt{1 + \left| \frac{1}{N(i)} \sum_{K=1}^{N(i)} \frac{\nabla u_i^K - \nabla v_i^K}{2} \right|^2} = \sqrt{4N(i)^2 + |\nabla m|^2}$$

17.9

Therefore:

$$P(E) \geq \sum_{i=1}^{M} \int_{G_i} \sqrt{4N(i)^2 + |\nabla m(z)|^2}\, dz$$

and

$$P(E^s) = \sum_{i=1}^{M} \int_{G_i} \sqrt{4 + |\nabla m(z)|^2}\, dz$$

Thus; since $N(i) \geq 1$:

$$\boxed{P(E^s) \leq P(E)}$$

Recall our notation:

$$D = \{z \in G : E_z \text{ is not an interval}\}$$

$$\therefore \; N(i) \geq 2 \iff G_i \cap D \neq \emptyset$$

Then:

$$P(E) - P(E^s) \geq \sum_{i=1}^{M} \int_{G_i \cap D} \sqrt{4N(i)^2 + |\nabla m|^2} - \sqrt{4 + |\nabla m|^2}\, dz$$

$$= \sum_{i=1}^{M} \int_{G_i \cap D} \frac{4(N(i)^2 - 1)}{\sqrt{4N(i)^2 + |\nabla m|^2} + \sqrt{4 + |\nabla m|^2}}\, dz$$

$$\geq 2 \sum_{i=1}^{M} \int_{G_i \cap D} \frac{1}{\sqrt{4N(i)^2 + |\nabla m|^2}} \; ; \quad \text{since } N(i) \geq 2$$

By Holder inequality:

$$2\mathcal{H}^{n-1}(D)^2 = 2\left(\int_D \frac{(4N(i)^2+|\nabla m|^2)^{1/4}}{(4N(i)^2+|\nabla m|^2)^{1/4}}\right)^2$$

$$\leq 2\left(\int_D \frac{1}{(4N(i)^2+|\nabla m|^2)^{1/2}}\right)\left(\int_D (4N(i)^2+|\nabla m|^2)^{1/2}\right)$$

$$= 2\left(\sum_{i=1}^{M}\int_{G_i\cap D} \frac{1}{\sqrt{4N(i)^2+|\nabla m|^2}}\right)\left(\sum_{i=1}^{M}\int_{D\cap G_i} \sqrt{4N(i)^2+|\nabla m|^2}\right)$$

$$\leq \left(P(E)-P(E^s)\right)\left(P(E)\right)$$

Thus, we have proved, for E a bounded set with polyhedral boundary and $\nu_E \cdot e_n \neq 0$ on ∂E that:

$$\boxed{\begin{array}{c} 2\mathcal{H}^{n-1}(D)^2 \leq P(E)\left(P(E)-P(E^s)\right) \\ P(E^s) \leq P(E) \end{array}}$$

which justifies (3) in the proof of Theorem 2. In conclusion, we have shown that if $E \subset \mathbb{R}^n$ is of finite perimeter, $|E|<\infty$ then E^s satisfies $P(E^s) \leq P(E)$ and, if $P(E^s)=P(E)$, then E_z is equivalent to an interval, for a.e. z. The rest of the proof of Theorem 2 can be found in textbook. ∎

This is another isoperimetric inequality that is not sharp.

Proposition (A perimeter bound on volume): If E is a bounded set of finite perimeter in $\mathbb{R}^n$, $n \geq 2$, then

$$P(E) \geq |E|^{\frac{n-1}{n}}$$

Proof: Following as in the proof of the Sobolev Imbedding Theorem (See "Modern Real Analysis", chapter 11) we have:

$$\|u\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq \|\nabla u\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)} \quad \forall u \in C_c^\infty(\mathbb{R}^n)$$

We now define:

$$u_\varepsilon = \chi_E * \rho_\varepsilon .$$

Recall that:

$$\int_{\mathbb{R}^n} |\nabla u_\varepsilon| \longrightarrow P(E) \quad \text{as } \varepsilon \to 0$$

Therefore:

$$P(E)^{\frac{n}{n-1}} = \lim_{\varepsilon \to 0} \|\nabla u_\varepsilon\|^{\frac{n}{n-1}}_{L^1(\mathbb{R}^n;\mathbb{R}^n)}$$

$$\geq \liminf_{\varepsilon \to 0} \int_{\mathbb{R}^n} |u_\varepsilon|^{\frac{n}{n-1}} \; ; \quad \text{by above Sobolev inequality}$$

$$\geq \int_{\mathbb{R}^n} \liminf_{\varepsilon \to 0} |u_\varepsilon|^{\frac{n}{n-1}} \; ; \quad \text{by Fatou's Lemma}$$

$$= \int_{\mathbb{R}^n} \chi_E = |E| . \quad \blacksquare$$

We look at the following application of isoperimetric inequalities:

Cheeger Sets : Let $p>0$, $n \geq 2$

A open set in $\mathbb{R}^n$.

The p-Cheeger problem in A is the variational problem:

$$c(p,A) = \inf \left\{ \frac{P(E)}{|E|^p}, \; E \subset A \right\} \qquad (**)$$

A minimizer E of $(**)$ is called a p-Cheeger set of A.

- If $p < \frac{n-1}{n}$ $\Rightarrow$ by scaling $c(p,A)=0$ and hence p-cheeger sets can not exist.

- If $p > \frac{n-1}{n}$ and A is bounded then p-Cheeger sets exist (Use the Direct method and the isoperimetric inequality $|E|^{\frac{n-1}{n}} \leq P(E)$).

- If $p = \frac{n-1}{n}$ then by the Isoperimetric inequality (Theorem 1 in Lecture 16) it follows that balls contained in A are the (only) p-Cheeger sets in A.