Lecture 18

The reduced boundary $\partial^* E$ of a set of locally finite perimeter E in $\mathbb{R}^n$ is the set of those $x \in \operatorname{spt} \mu_E$ such that:

$$\lim_{r\to 0^+} \frac{\mu_E(B(x,r))}{|\mu_E|(B(x,r))} \quad \text{exists and belongs to } S^{n-1}.$$

From Lebesgue-Besicovitch differentiation theorem we know that the above limit exists for $|\mu_E|$-a.e. $x \in \mathbb{R}^n$. Indeed, since $\mu_E << |\mu_E|$ then:

$$\mu_E = \left(D_{|\mu_E|}\mu_E\right)|\mu_E|, \quad D_{|\mu_E|}\mu_E(x) = \lim_{r\to 0} \frac{\mu_E(B(x,r))}{|\mu_E|(B(x,r))}$$

Define:

$$\nu_E(x) := D_{|\mu_E|}\mu_E(x) = \lim_{r\to 0}\frac{\mu_E(B(x,r))}{|\mu_E|(B(x,r))}, \quad \text{whenever } x \in \partial^* E,$$

i.e. when x is such that the limit exists and $|\nu_E| = 1$

<u>Remark 1</u>: If $|E \Delta F| = 0$ then, since $\mu_E = \mu_F$ we conclude that $\partial^* E = \partial^* F$

From the Riesz Representation Theorem we have that:

$$|D_{|\mu_E|}\mu_E| = 1 \text{ for } |\mu_E|\text{-a.e. } x \in \mathbb{R}^n.$$

Therefore, we have:

$$\boxed{\mu_E = \nu_E |\mu_E| \llcorner \partial^* E},$$ ν_E is the measure-theoretic outer unit normal to E.

So that the distributional Gauss-Green theorem takes the form:

$$\int_E \nabla \varphi = \int_{\partial^* E} \varphi \, \nu_E \, d|\mu_E|.$$

By definition:

$\partial^* E \subset \operatorname{spt} \mu_E$. By definition of spt μ_E we also have $\overline{\partial^* E} \subset \operatorname{spt} \mu_E$

Recall:

$$\operatorname{spt} \mu_E \subset \partial E$$

$$\therefore \quad \boxed{\overline{\partial^* E} \subset \partial E} \quad (1)$$

Note that $|\mu_E|(\mathbb{R}^n \setminus \partial^* E) = 0$ and hence $|\mu_E|$ is concentrated on $\partial^* E$. Note that $|\mu_E|$ is also concentrated on $\overline{\partial^* E}$ because:

$|\mu_E|(\mathbb{R}^n \setminus \overline{\partial^* E}) = 0$.

By definition of support of a measure:

$$\operatorname{spt} \mu_E = \bigcap_{\substack{F \text{ closed} \\ |\mu_E|(\mathbb{R}^n \setminus F) = 0}}$$

Since $F = \overline{\partial^* E}$ is a candidate in this intersection then:

$$\boxed{\operatorname{spt} \mu_E \subset \overline{\partial^* E}} \qquad (2)$$

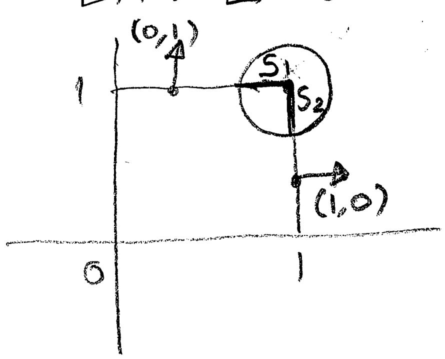
Recall, we proved in a previous lecture that we can modify E on a set of measure zero to have:

$$\operatorname{spt} \mu_E = \partial E.$$

With such E, from (1) and (2) we obtain

$$\boxed{\overline{\partial^* E} = \partial E}$$

Ex: $E = [0,1] \times [0,1] \subset \mathbb{R}^2$



$$\frac{\mu_E(B(x,r))}{|\mu_E|(B(x,r))} = \frac{\int_{B(x,r)\cap S_1} (0,1)\, d\mathcal{H}^{n-1} + \int_{B(x,r)\cap S_2} (1,0)\, d\mathcal{H}^{n-1}}{\int_{B(x,r)\cap \partial E} d\mathcal{H}^{n-1}}$$

$x = (1,1)$

$$= \frac{\frac{1}{r}(0,1) + \frac{1}{r}(1,0)}{2r}$$

$$= \frac{\frac{1}{r}(1,1)}{2r} = \left(\frac{1}{2}, \frac{1}{2}\right)$$

Hence $\left|\left(\frac{1}{2}, \frac{1}{2}\right)\right| < 1$ and the corner is not in $\partial^* E$

In this example we have used that if E is a set with Lipschitz boundary (or C^1) then:

$$\mu_E = \nu_E \mathcal{H}^{n-1} \llcorner \partial E, \qquad |\mu_E| = \mathcal{H}^{n-1} \llcorner \partial E$$

where ν_E is the classical outer unit normal to E. If E has C^1 boundary, from $\mu_E = \nu_E |\mu_E| \llcorner \partial^* E$, we get that:

$$\partial^* E = \partial E; \quad \nu_E = D_{|\mu_E|}\mu_E$$ is the classical outer unit normal.

We now proceed to show two main theorems about $\partial^* E$:

Theorem 1: About tangential properties of the reduced boundary

and

Theorem 2: About the structure of a set of finite perimeter saying that $\partial^* E$ is a rectifiable set.

In order to accomplish this, we will study the blow-ups of E:

$$E_{x,r} = \frac{E-x}{r} = \Phi_{x,r}(E), \quad x \in \mathbb{R}^n, \ r>0$$

where $\Phi_{x,r}(y) = \frac{y-x}{r}$

We now start the study of

Theorem 1: (Tangential properties of the reduced boundary). If E is a set of locally finite perimeter in $\mathbb{R}^n$, and $x \in \partial^* E$, then

$$E_{x,r} \xrightarrow{loc} H_x = \{ y \in \mathbb{R}^n : y \cdot \nu_E \leq 0 \}, \text{ as } r \to 0^+.$$

Similarly, if $\Pi_x = \partial H_x = \nu_E(x)^{\perp}$, then, as $r \to 0^+$:

$$\mu_{E_{x,r}} \xrightarrow{*} \nu_E(x) \mathcal{H}^{n-1} \llcorner \Pi_x, \quad |\mu_{E_{x,r}}| \xrightarrow{*} \mathcal{H}^{n-1} \llcorner \Pi_x$$

Remark 2: We will prove in Theorem 2 that

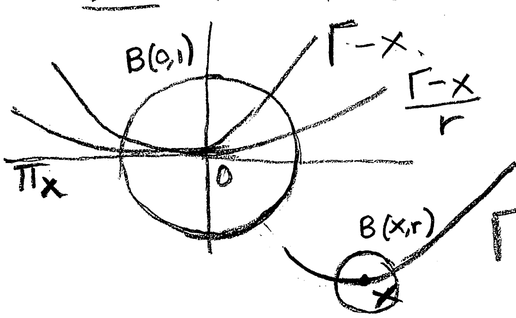
$$\mu_E = \nu_E \mathcal{H}^{n-1} \llcorner \partial^* E,$$

and then we can write Theorem 1 in a more expressive form, namely:

$$\nu_E \mathcal{H}^{n-1} \llcorner \left(\frac{\partial^* E - x}{r} \right) \xrightarrow{*} \nu_E(x) \mathcal{H}^{n-1} \llcorner \Pi_x$$

$$\mathcal{H}^{n-1} \llcorner \left(\frac{\partial^* E - x}{r} \right) \xrightarrow{*} \mathcal{H}^{n-1} \llcorner \Pi_x$$

Ex: Recall our example when Γ is a smooth curve:



$r < 1$

$$(\Phi_{x,r})_\# (\mathcal{H}^1 \llcorner \Gamma)(B(0,1)) = \mathcal{H}^1 \llcorner \Gamma (B(x,r))$$

$$\frac{1}{r} (\Phi_{x,r})_\# (\mathcal{H}^1 \llcorner \Gamma) = \mathcal{H}^1 \llcorner \frac{\Gamma - x}{r} \xrightarrow{*} \mathcal{H}^1 \llcorner \Pi_x$$

$$\left[\frac{1}{r} (\Phi_{x,r})_\# (\mathcal{H}^1 \llcorner \Gamma) \right] (B(0,1)) \xrightarrow{r \to 0} \mathcal{H}^1 \llcorner \Pi_x (B(0,1))$$

Corollary of Theorem 1:

If E is a set of locally finite perimeter and $x \in \partial^* E$, then:

$$\lim_{r \to 0^+} \frac{|E \cap B(x,r)|}{\omega_n r^n} = \frac{1}{2}$$

$$\lim_{r \to 0^+} \frac{P(E\,;\,B(x,r))}{\omega_{n-1} r^{n-1}} = 1.$$

In particular, $\partial^* E \subset E^{(1/2)}$, the set of points of density one-half of E.

Proof: Let $x \in \partial^* E$. Since $E_{x,r} \xrightarrow{loc} H_x$, then:

$$|E_{x,r} \cap B(0,1)| \to |H_x \cap B(0,1)| \quad \text{as } r \to 0$$

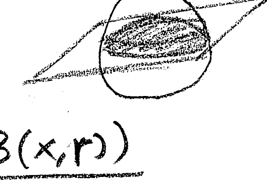
$$\shortparallel$$

$$\frac{|E \cap B(x,r)|}{r^n}$$

$$\therefore \frac{|E \cap B(x,r)|}{\omega_n r^n} \to \frac{|H_x \cap B(0,1)|}{\omega_n r^n} = \frac{1}{2}$$

Since $|\mu_{E_{x,r}}| \xrightarrow{*} \mathcal{H}^{n-1} \llcorner \Pi_x$, and since $\mathcal{H}^{n-1}(\Pi_x \cap \partial B(0,1)) = 0$, we have:

$$|\mu_{E_{x,r}}|(B(0,1)) \to \mathcal{H}^{n-1} \llcorner \Pi_x (B(0,1))$$



By Lemma 1 below, $|\mu_{E_{x,r}}|(B(0,1)) = \dfrac{P(E\,;\,B(x,r))}{r^{n-1}}$

$$\therefore \frac{P(E\,;\,B(x,r))}{r^{n-1}} \to \omega_{n-1}, \text{ i.e. } \frac{P(E\,;\,B(x,r))}{\omega_{n-1} r^{n-1}} \to 1. \quad \square$$

Lemma 1: If E is a set of locally finite perimeter in $\mathbb{R}^n$, $x \in \mathbb{R}^n$, $r>0$ then $E_{x,r}$ is a set of locally finite perimeter in $\mathbb{R}^n$ with

$$\mu_{E_{x,r}} = \frac{(\Phi_{x,r})_{\#}\,\mu_E}{r^{n-1}}$$

18.7

Proof: First, notice that:

$$\frac{(\Phi_{x,r})_{\#}\,\mu_E}{r^{n-1}}(B(0,1)) = \frac{\mu_E(B(x,r))}{r^{n-1}} = \frac{P(E; B(x,r))}{r^{n-1}},$$

which we used in previous corollary. Now, let $\varphi \in C_c^1(\mathbb{R}^n)$ and $\varphi_{x,r} = \varphi \circ \Phi_{x,r}$. Thus,

$$\nabla \varphi_{x,r}(y) = \nabla\left(\varphi\left(\frac{y-x}{r}\right)\right)$$
$$= \nabla\varphi\left(\frac{y-x}{r}\right)\cdot\frac{1}{r}$$
$$= \frac{1}{r}\nabla\varphi \circ \Phi_{x,r}(y),$$

and

$$\int_{E_{x,r}} \nabla\varphi \underset{\uparrow}{=} \frac{1}{r^n}\int_E \nabla\varphi \circ \Phi_{x,r} = \frac{1}{r^{n-1}}\int_E \frac{1}{r}\nabla\varphi\circ\Phi_{x,r}$$

Area formula

$$= \frac{1}{r^{n-1}}\int_E \nabla\varphi_{x,r}$$

$$= \frac{1}{r^{n-1}}\int_{\mathbb{R}^n} \varphi_{x,r}\, d\mu_E \, ;$$

Recall $\int_E \nabla\phi = \int_{\mathbb{R}^n} \phi \, d\mu_E$, $\forall \phi \in C_c(\mathbb{R}^n)$. Use $\phi = \varphi_{x,r}$

$$= \frac{1}{r^{n-1}} \int_{\mathbb{R}^n} \varphi \, d(\Phi_{x,r})_{\#} \mu_E \;;$$

since the push forward satisfies:

$$\int_{\mathbb{R}^m} u \, d(f_{\#}\mu) = \int_{\mathbb{R}^n} (u \circ f) \, d\mu \,, \qquad f: \mathbb{R}^n \to \mathbb{R}^m$$

μ Radon measure on $\mathbb{R}^n$.

We have shown:

$$\int_{E_{x,r}} \nabla \varphi = \frac{1}{r^{n-1}} \int_{\mathbb{R}^n} \varphi \, d(\Phi_{x,r})_{\#} \mu_E \quad \forall \varphi \in C_c^1(\mathbb{R}^n)$$

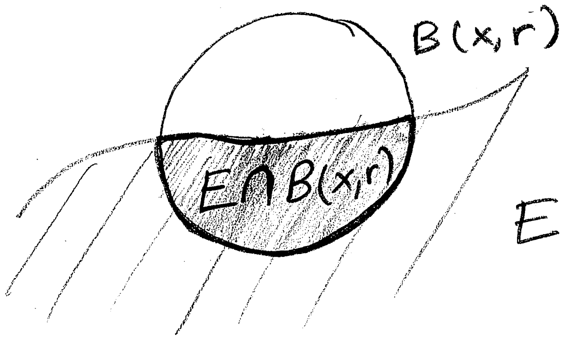
Thus, since $\frac{1}{r^{n-1}} (\Phi_{x,r})_{\#} \mu_E$ is a Radon measure, $E_{x,r}$ is a set of locally finite perimeter, with

$$\boxed{\mu_{E_{x,r}} = \frac{1}{r^{n-1}} (\Phi_{x,r})_{\#} \mu_E}$$

In order to prove Theorem 1, we need several Lemmas, which we now state without proof (please refer to our textbook for the proofs.

Lemma 2 (Intersection with a ball). $E \subset \mathbb{R}^n$ set of locally finite perimeter. Then, for every $r>0$, $E \cap B(x,r)$ is a set of finite perimeter in $\mathbb{R}^n$. Moreover, for a.e. $r>0$:

- $\mu_{E\cap B(x,r)} = \nu_{B(x,r)} \mathcal{H}^{n-1} \llcorner (E \cap \partial B(x,r)) + \mu_E \llcorner B(x,r)$
- $|\mu_{E\cap B(x,r)}| = \mathcal{H}^{n-1} \llcorner (E \cap \partial B(x,r)) + |\mu_E| \llcorner B(x,r)$
- $P(E \cap B(x,r)) = \mathcal{H}^{n-1}(E \cap \partial B(x,r)) + P(E; B(x,r))$



Lemma 2 is geometrically very intuitive since, for $\mathcal{H}^{n-1}$-a.e. r, E and $B(x,r)$ intersect transversally and thus $\mathcal{H}^{n-1}(\partial E \cap \partial B(x,r)) = 0$. Hence, the perimeter of the intersection "has two pieces", one being $\partial E \cap \partial B(x,r)$ and the other $\partial B(x,r) \cap E$. The rigorous proof is based on the generalized divergence theorem $\int_E \operatorname{div} T = \int_E T \cdot d\mu_E$, $\forall T \in C_c^1(\mathbb{R}^n)$ and the approximation of E by smooth open sets with C^1-boundary proved in a previous lecture.

<u>Lemma 3</u>: Characterizations of half-spaces: If F is a set of locally finite perimeter in $\mathbb{R}^n$ and $\nu \in S^{n-1}$ is such that:

$$\nu_F(y) = \nu \quad \text{for } |\mu_F|\text{-a.e } y \in \partial^* F$$

then $\exists \alpha \in \mathbb{R}$ such that F is equivalent to the half space $\{z \in \mathbb{R}^n : z \cdot \nu < \alpha\}$.

Proof of Theorem 1:

Let $x \in \partial^* E$. By Lemma 2,

(a) $\mu_{E\cap B(x,r)} = \nu_{B(x,r)} \mathcal{H}^{n-1} \llcorner (E \cap \partial B(x,r)) + \mu_E \llcorner B(x,r)$

(b) $P(E\cap B(x,r)) = \mathcal{H}^{n-1}(E \cap \partial B(x,r)) + P(E; B(x,r))$

for a.e. $r>0$.

<u>Step one</u>: $\exists r(x)$ and $c(n)>0$ s.t:

$P(E\cap B(x,r)) \leq 3\mathcal{H}^{n-1}(E\cap\partial B(x,r))$ for a.e. $r<r(x)$

$P(E; B(x,r)) \leq c(n)\, r^{n-1}$ $\forall r<r(x)$

Let $\varphi \in C_c^1(\mathbb{R}^n)$, $\varphi \equiv 1$ on $\overline{B}(x,r)$

$$0 = \int_{E\cap B(x,r)} \nabla\varphi = \int_{\mathbb{R}^n} \varphi \, d\mu_{E\cap B(x,r)}$$

$$= \int_{E\cap\partial B(x,r)} \varphi \nu_{B(x,r)} \, d\mathcal{H}^{n-1} + \int_{B(x,r)} \varphi \, d\mu_E$$

$$= \int_{E\cap\partial B(x,r)} \nu_{B(x,r)} \, d\mathcal{H}^{n-1} + \mu_E(B(x,r))$$

$\therefore$ $|\mu_E(B(x,r))| \leq \mathcal{H}^{n-1}(E\cap\partial B(x,r))$.

Now,

$x \in \partial^* E \Rightarrow \dfrac{|\mu_E(B(x,r))|}{|\mu_E|(B(x,r))} \to 1 \Rightarrow \exists r(x)$ s.t: $|\mu_E|(B(x,r)) \leq 2\,|\mu_E(B(x,r))|$ $\forall r \leq r(x)$

Hence:

$P(E; B(x,r)) \leq 2\mathcal{H}^{n-1}(E\cap\partial B(x,r))$

$\Rightarrow$ $P(E\cap B(x,r)) \leq 3\mathcal{H}^{n-1}(E\cap\partial B(x,r))$; by (b).

Trivially,

$$\mathcal{H}^{n-1}(E \cap \partial B(x,r)) \leq n\omega_n r^{n-1}$$

$$\Rightarrow \quad P(E; B(x,r)) \leq 2\mathcal{H}^{n-1}(E \cap \partial B(x,r))$$
$$\leq n\omega_n r^{n-1} \quad , \text{ for a.e. } r>0,$$

but actually this inequality holds for every $r<r(x)$ since $r \longmapsto P(E; B(x,r))$ is an increasing function. This completes the proof of step 1.

We will finish the proof of Theorem 1 next class.