Lecture 19

Continuation of proof of Theorem 1 (Tangential properties of the reduced boundary). See Lecture 18.

<u>Step two</u>: We prove two lower bounds on the n-density ratios of E and $\mathbb{R}^n\setminus E$ at $x\in\partial^*E$:

$$\frac{|E\cap B(x,r)|}{r^n}\geq\frac{1}{(3n)^n}\quad \forall r<r(x) \qquad (1)$$

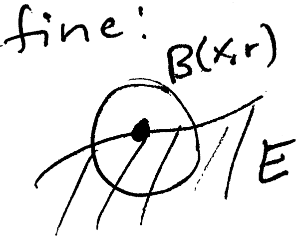
$$\frac{|(\mathbb{R}^n\setminus E)\cap B(x,r)|}{r^n}\geq\frac{1}{(3n)^n},\quad \forall r<r(x). \qquad (2)$$

Recall that E and $\mathbb{R}^n\setminus E$ satisfy:

$$\mu_E=-\mu_{\mathbb{R}^n\setminus E},$$

and hence
$$\partial^*E=\partial^*(\mathbb{R}^n\setminus E).$$

Thus, we only need to prove (1). Define:



$$m(r)=|E\cap B(x,r)|,\ r>0$$

By coarea formula:
$$m(r)=|E\cap B(x,r)|=\int_0^r\mathcal{H}^{n-1}(E\cap\partial B(x,t))\,dt. \qquad (3)$$

From (3) we have:
m is absolutely continuous and:
$$m'(r)=\mathcal{H}^{n-1}(E\cap\partial B(x,r))\ \text{for a.e. } r>0.$$

Note that $m(r)>0$, $r>0$, $m(0)=0$.

Indeed if $m(r)=0$ for some $r>0$

where the following holds:

$$P(E\cap B(x,r)) = \mathcal{H}^{n-1}(E\cap\partial B(x,r)) + P(E; B(x,r)),$$

then, since $m(r)=0 \Rightarrow P(E\cap B(x,r))=0$. Then, for the equality to hold we need:

$$P(E; B(x,r))=0,$$

but $x\in\partial^* E \Rightarrow x\in \operatorname{spt}\mu_E \Rightarrow |\mu_E|(B(x,r))>0$
$\Rightarrow P(E; B(x,r))>0$. We conclude that $m(r)>0$ for a.e. r, but since m is continuous and increasing, $m(r)>0$, $\forall r>0$. Then

$$m(r)^{\frac{n-1}{n}} = |E\cap B(x,r)|^{\frac{n-1}{n}}$$

$$\leq P(E\cap B(x,r)); \quad \text{See Lecture 17, Page 17.11}$$

$$\leq 3\mathcal{H}^{n-1}(E\cap\partial B(x,r)); \quad \text{by Step 1, for a.e. } r<r(x)$$

$$= 3\, m'(r)$$

$$\therefore \quad m(r)^{\frac{-n+1}{n}} m'(r) \geq \frac{1}{3}$$

$$\therefore \quad m(r)^{-1+\frac{1}{n}} m'(r) \geq \frac{1}{3}$$

$$\therefore \quad \frac{d}{dr}\left(n\, m(r)^{1/n}\right) \geq \frac{1}{3} \quad \text{a.e. } r<r(x)$$

Integrating both sides:

$$n\, m(r)^{1/n} \geq \frac{1}{3} r, \qquad \forall r < r(x)$$

$$\Rightarrow \quad m(r)^{1/n} \geq \frac{r}{3n}$$

$$\Rightarrow \quad m(r) \geq \frac{r^n}{(3n)^n} \quad \Rightarrow \quad \frac{|E \cap B(x,r)|}{r^n} \geq \frac{1}{(3n)^n}, \quad r < r(x)$$

Step three:

To prove:

$$E_{x,r} \xrightarrow{loc} H_x, \text{ as } r \to 0^+,$$

it is enough to show that for every $\{r_i\}$, $r_i \to 0$, $\exists \{r_{i_k}\}$ such that

$$E_{x, r_{i_k}} \xrightarrow{loc} H_x, \quad \text{as } K \to \infty$$

Note:

$$P(E_{x,r}; B_R) = \frac{P(E; B(x, rR))}{r^{n-1}}$$; we proved in Lecture 18 that $|\mu_{E_{x,r}}| = \frac{1}{r^{n-1}} (\Phi_{x,r})_{\#} |\mu_E|$

$$\leq \frac{C(n)(rR)^{n-1}}{r^{n-1}}; \quad \forall rR < r(x) \text{ by step one}$$

$$= C(n) R^{n-1}; \quad \forall r < \frac{r(x)}{R}$$

By compactness, (Lecture 14, Page 14.1), $\exists$ F set of locally finite perimeter, and a subsequence $\{E_{x,r_{i_j}}\}$ such that

$$E_{x,r_{i_j}} \xrightarrow[j\to\infty]{loc} F, \qquad \mu_{E_{x,r_{i_j}}} \xrightarrow[j\to\infty]{*} \mu_F$$

For simplicity, we denote $\{E_{x,r_{i_j}}\}_{j=1}^{\infty}$ again as $\{E_{x,r_i}\}$. Now, up to extracting a further subsequence:
$\exists \lambda$ such that $|\mu_{E_{x,r_i}}| \xrightarrow{*} \lambda$. Then

(4) $$\boxed{\lim_{i\to\infty} \mu_{E_{x,r_i}}(B_R) \to \mu_F(B_R) \quad \text{a.e. } R>0 \text{ where } \lambda(\partial B_R)=0}$$

Also, since $x \in \partial^* E$ and $|\mu_{E_{x,r}}| = \frac{1}{r^{n-1}}(\Phi_{x,r})_\# |\mu_E|$:

$$\lim_{r\to 0^+} \frac{\mu_{E_{x,r}}(B_R)}{|\mu_{E_{x,r}}|(B_R)} = \lim_{r\to 0^+} \frac{\mu_E(B(x,rR))}{|\mu_E|(B(x,rR))}$$

$$= \nu_E(x)$$

$$\therefore \lim_{r\to 0^+} \frac{P(E_{x,r};B_R)}{\nu_E(x)\cdot\mu_{E_{x,r}}(B_R)} = 1 \, ;$$ Taking dot product with $\nu_E(x)$ on both sides.

We have:

$$P(F;B_R) \le \liminf_{i\to\infty} P(E_{x,r_i};B_R) \; ; \text{ lower semicontinuity}$$

$$= \lim_{i\to\infty} \nu_E(x)\cdot\mu_{E_{x,r}}(B_R)$$

$$= \nu_E(x)\cdot \mu_F(B_R) \; ; \quad \text{by } (4)$$

$$\leq |\mu_F(B_R)|$$

$$\leq |\mu_F|(B_R) = P(F; B_R)$$

We have shown; for a.e. R:

$$\boxed{\begin{array}{l} |\mu_F|(B_R) = \nu_E(x)\cdot \mu_F(B_R) \\ |\mu_F|(B_R) = \lim\limits_{i\to\infty} |\mu_{E_{x,r_i}}|(B_R) \end{array}} \quad (5)$$

By (5) and (4) we have:

$$\boxed{|\mu_{E_{x,r_i}}| \xrightarrow{*} |\mu_F| \; ;} \quad (6)$$

(6) is true by the fact that (See Exercise 4.31):

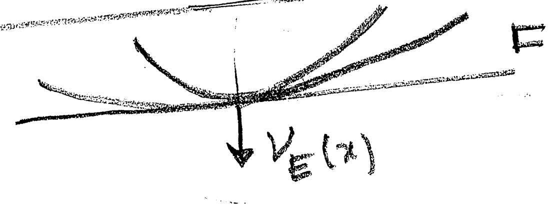
$$\boxed{\mu_k \xrightarrow{*} \mu \text{ and } |\mu_k|(B_{r_j}) \underset{k\to\infty}{\rightarrow} |\mu|(B_{r_j}) \; \forall j, \; \overset{\text{some}}{r_j\to\infty} \Rightarrow |\mu_k| \xrightarrow{*} |\mu|}$$

By (5):

$$0 = \int_{B_R} \overbrace{(1-\nu_E(x)\cdot\nu_F(y))}^{\geq 0}\, d|\mu_F|(y) \; , \text{ for a.e. } R>0$$

$$\therefore \; 1 - \nu_E(x)\cdot\nu_F(y) = 0 \quad \text{for } |\mu_F|\text{-a.e. } y\in\partial^*F$$

$$\therefore \; \boxed{\nu_E(x) = \nu_F(y) \;\; \text{for } |\mu_F|\text{-a.e. } y\in\partial^*F} \quad (7)$$



By Lemma 3, Lesson 18 we have that F is equivalent to a half space. That is, $\exists \alpha \in \mathbb{R}$ such that:

$$|F \,\Delta\, \{y \in \mathbb{R}^n : \nu_E(x)\cdot y < \alpha\}| = 0.$$

We have two possibilities:

- $\alpha < 0$. In this case $|F \cap B_\alpha| = 0$, so that

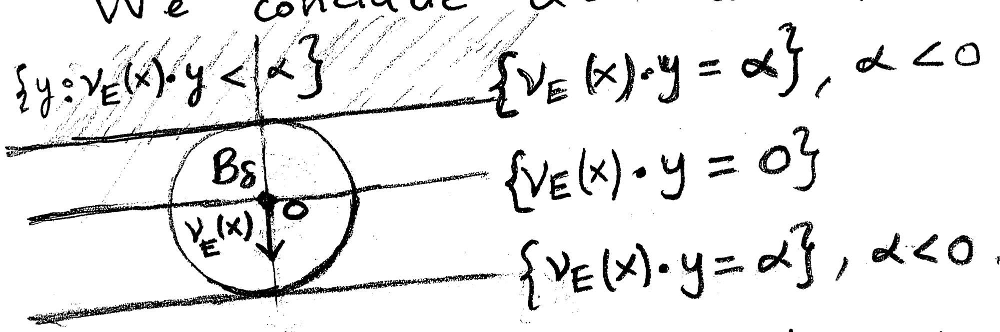
$$0 = \frac{|F \cap B_\alpha|}{|B_\alpha|} = \lim_{i\to\infty} \frac{|E_{x,r_i} \cap B_\alpha|}{|B_\alpha|}, \quad E_{x,r_i} \xrightarrow{loc} F$$

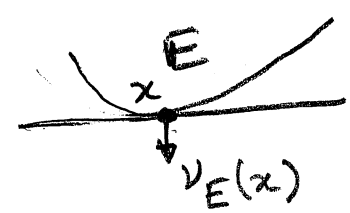
$$= \lim_{i\to\infty} \frac{|E \cap B(x, r_i\alpha)|}{|B(x, r_i\alpha)|},$$

which is a contradiction to Step 2.

- $\alpha > 0$. The same argument gives a contradiction.

We conclude $\alpha = 0$ and hence $F = H_x$.





Step four: We have proved that as $r \to 0$:

$$\mu_{E_{x,r}} \xrightarrow{*} \mu_{H_x}, \quad E_{x,r} \xrightarrow{loc} H_x, \quad |\mu_{E_{x,r}}| \xrightarrow{*} |\mu_{H_x}|$$

But by the Gauss-Green Theorem:

$$\mu_{H_x} = \nu_E(x)\, \mathcal{H}^{n-1} \llcorner \partial H_x,$$

which concludes the proof of Theorem 1. ∎

Let us now make a resume of important things we have learned so far:

- $M \subset \mathbb{R}^n$, $\mathcal{H}^k \llcorner M$ Radon measure. M is locally $\mathcal{H}^k$-rectifiable if $\exists \{f_i\}$, $f_i : \mathbb{R}^k \to \mathbb{R}^n$ Lipschitz, such that:

$$\mathcal{H}^k\left(M \setminus \bigcup_{i=1}^{\infty} f_i(\mathbb{R}^k)\right) = 0$$

- <u>Criterion for rectifiability</u>: M Borel set, μ Radon measure on $\mathbb{R}^n$, $\mu(\mathbb{R}^n \setminus M) = 0$. If $\forall x \in M$, $\exists \Pi_x$ a k-dimensional hyperplane such that

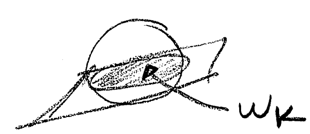
$$\frac{1}{r^k} (\Phi_{x,r})_\# \mu \xrightarrow{*} \mathcal{H}^k \llcorner \Pi_x ,$$

then M is locally $\mathcal{H}^k$-rectifiable and:

$$\mu = \mathcal{H}^k \llcorner M .$$

Hypothesis can be written as (recall $\Phi_{x,r}(y) = \frac{y-x}{r}$):

$$\frac{\mu(B(x,r))}{r^k} \longrightarrow \omega_k ;$$



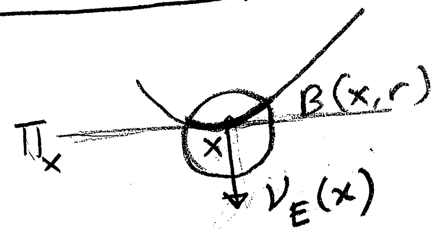
as $r \to 0$, or $\frac{\mu(B(x,r))}{\omega_k r^k} \to 1$. Indeed;

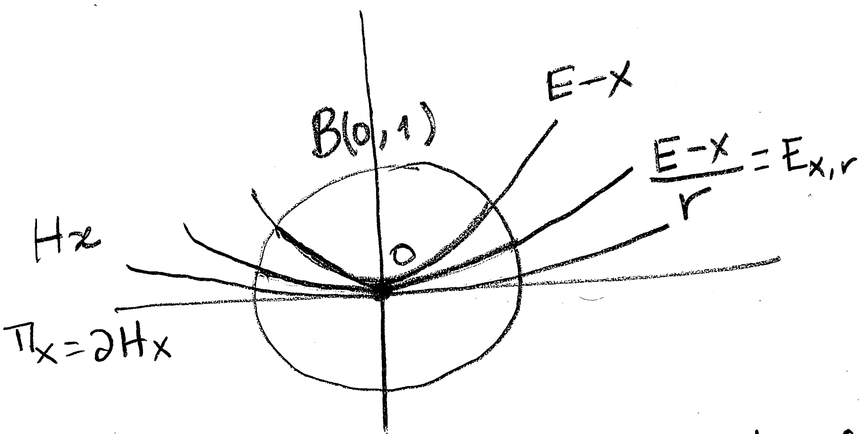
the weak convergence implies (since $\mathcal{H}^k \llcorner \Pi_x (\partial B(0,1)) = 0$):

$$\frac{1}{r^k} (\Phi_{x,r})_\# \mu \,(B(0,1)) \to \mathcal{H}^k \llcorner \Pi_x (B(0,1)) ,$$

and $(\Phi_{x,r})_\# \mu \,(B(0,1)) = \mu\left(\Phi_{x,r}^{-1}(B(0,1))\right) = \mu(B(x,r))$

• Sets of finite perimeter:





$$\boxed{\mu_{E_{x,r}} = \frac{1}{r^{n-1}} \left(\Phi_{x,r}\right)_\# \mu_E}$$

Lemma 1, Lecture 18.

Note: In the proof of Lemma 1 we have shown that:

$$\int_{E_{x,r}} \nabla\varphi = \frac{1}{r^{n-1}} \int_{\mathbb{R}^n} \varphi_{x,r}\, d\mu_E, \quad \forall \varphi \in C_c^1(B(0,R)), \quad \varphi_{x,r} = \varphi\left(\frac{y-x}{r}\right)$$

But $\varphi \in C_c^1(B(0,R)) \iff \varphi_{x,r} \in C_c^1(B(x,rR))$. Thus, taking the sup over all such φ:

$\Rightarrow |\mu_{E_{x,r}}|(B(0,R)) = \frac{1}{r^{n-1}} |\mu_E|(B(0,rR))$; that is: $|\mu_{E_{x,r}}| = \frac{1}{r^{n-1}} (\Phi_{x,r})_\# |\mu_E|$.

• Theorem 1 says (Lecture 18, Page 18.5) if $x \in \partial^* E$

then: (a) $E_{x,r} \xrightarrow{loc} H_x$

(b) $|\mu_{E_{x,r}}| \xrightarrow{*} \mathcal{H}^{n-1} \llcorner \Pi_x$, $\mu_{E_{x,r}} \xrightarrow{*} \nu_E(x) \mathcal{H}^{n-1} \llcorner \Pi_x$

We proved (Page 18.6) that:

(a) implies $\dfrac{|E \cap B(x,r)|}{|B(x,r)|} \longrightarrow \dfrac{1}{2}$ $\therefore$ $\partial^* E \subset E^{1/2}$

(b) implies $|\mu_{E_{x,r}}|(B(0,1)) = \frac{1}{r^{n-1}} |\mu_E|(B(x,r)) = \dfrac{P(E;B(x,r))}{r^{n-1}} \longrightarrow \omega_{n-1}$

or $\boxed{\dfrac{P(E;B(x,r))}{\omega_{n-1} r^{n-1}} \longrightarrow 1}$; which is the same as:

$$\boxed{|\mu_{E_{x,r}}|(B(0,1)) \xrightarrow{r \to 0} \omega_{n-1}}$$

- Since

$$\frac{|\mu_E|(B(x,r))}{\omega_{n-1} r^{n-1}} \xrightarrow{r\to 0} 1 \quad \forall x \in \partial^* E,$$

the Criterion for rectifiability implies that

$$\boxed{\begin{array}{l} \partial^* E \text{ is locally } \mathcal{H}^{n-1}\text{-rectifiable} \\ \text{and} \quad |\mu_E| = \mathcal{H}^{n-1} \llcorner \partial^* E \end{array}}$$

Since we now that $\mu_E = \nu_E |\mu_E| \llcorner \partial^* E$, we obtain:

$$\boxed{\mu_E = \nu_E \mathcal{H}^{n-1} \llcorner \partial^* E}$$

We have proved the following Corollary of Theorem 1:

<u>Corollary</u>: If E is a set of (locally) finite perimeter, then $\partial^* E$ is (locally) $\mathcal{H}^{n-1}$-rectifiable and

$$\mu_E = \nu_E \mathcal{H}^{n-1} \llcorner \partial^* E.$$

Moreover, the approximate tangent space to $\partial^* E$ at $x \in \partial^* E$ agrees with the orthogonal space to the measure-theoretic outer unit normal to E at x, that is,

$$T_x(\partial^* E) = \nu_E(x)^{\perp}.$$

Federer's Theorem

We already now that:

$$\partial^* E \subset E^{(1/2)}$$

<u>Definition</u>: The essential boundary $\partial^e E$ of a Lebesgue measurable set $E \subset \mathbb{R}^n$ is defined as:

$$\partial^e E = \mathbb{R}^n \setminus (E^{(0)} \cup E^{(1)}).$$

Obviously:

$$E^{(1/2)} \subset \partial^e E.$$

We have

Theorem (<u>Federer's theorem</u>): $E \subset \mathbb{R}^n$ set of locally finite perimeter, then $\partial^* E \subset E^{(1/2)} \subset \partial^e E$, with

$$\mathcal{H}^{n-1}(\partial^e E \setminus \partial^* E) = 0$$

<u>Proof</u>: The relative isoperimetric inequality says (we will prove later, see chapter 12 from textbook):

$$P(E; B(x,r)) \geq c(n) \min\{|E \cap B(x,r)|, |B(x,r) \setminus E|\}^{\frac{n-1}{n}}$$

$$|E \cap B(x,r)| \leq \omega_n r^n \implies |E \cap B(x,r)|^{\frac{n-1}{n}} \leq \omega_n^{\frac{n-1}{n}} r^{n-1}$$

$$\implies |E \cap B(x,r)|^{1/n} \leq \omega_n^{1/n} r$$

$$\implies |E \cap B(x,r)|^{1/n} |E \cap B(x,r)|^{\frac{n-1}{n}} \leq \omega_n^{1/n} r \, |E \cap B(x,r)|^{\frac{n-1}{n}}$$

$$\therefore \quad \omega_n^{1/n} r \, |E \cap B(x,r)|^{\frac{n-1}{n}} \geq |E \cap B(x,r)|$$

Therefore,

$$\text{(A)}\quad \frac{P(E; B(x,r))}{r^{n-1}} \geq c(n) \min\left\{ \frac{|E \cap B(x,r)|}{r^n}, \frac{|B(x,r) \setminus E|}{r^n} \right\}$$

If $\Theta^*_{n-1}(\mathcal{H}^{n-1} \llcorner \partial^* E, x) = \limsup_{r \to 0} \dfrac{\mathcal{H}^{n-1}(\partial^* E \cap B(x,r))}{\omega_{n-1} r^{n-1}}$

$$= 0$$

then, by (A) above, $x \in E^{(0)} \cup E^{(1)}$. Thus,

$x \in \partial^e E \implies \Theta^*_{n-1}(\mathcal{H}^{n-1} \llcorner \partial^* E, x) > 0$

$$\therefore \partial^e E \subset \{ x \in \mathbb{R}^n : \Theta^*_{n-1}(\mathcal{H}^{n-1} \llcorner \partial^* E, x) > 0 \} \quad \text{(B)}$$

(Recall that $x \in \partial^* E \implies \Theta^*_{n-1}(\mathcal{H}^{n-1} \llcorner \partial^* E, x) = 1$).

From (B):

$\partial^e E \setminus \partial^* E \subset F := \{ x \in \mathbb{R}^n \setminus \partial^* E : \Theta^*_{n-1}(\mathcal{H}^{n-1} \llcorner \partial^* E)(x) > 0 \}$.

We now use a Corollary we proved in Lecture 7 (Page 7.4) and notice that we have already used this Corollary many times!

(Corollary: $M \subset \mathbb{R}^n$, $\mathcal{H}^s(M \cap K) < \infty \ \forall K$ compact. Then
$\lim_{r \to 0} \dfrac{\mathcal{H}^s(M \cap B(x,r))}{\omega_s r^s} = 0$ for $\mathcal{H}^s$-a.e. $x \in \mathbb{R}^n \setminus M$)

With $M = \partial^* E$ and $s = n-1$ we get:

$$\mathcal{H}^{n-1}(F) = 0$$

$$\therefore \mathcal{H}^{n-1}(\partial^e E \setminus \partial^* E) = 0 . \quad \blacksquare$$