<u>Problem</u>: $\mathcal{M}(\mu)$ could be "to small" to work with. This lead us to introduce:

<u>Def</u>: $\mathcal{B}(\mathbb{R}^n)$ = Borel sets = smallest σ-algebra containing the open sets.

<u>Def</u>: $\mu$ is a Borel measure if $\mu$ is an outer measure $\mu$ on $\mathbb{R}^n$ such that $\mathcal{B}(\mathbb{R}^n) \subset \mathcal{M}(\mu)$.

<u>Carathéodory criterion</u>: If $\mu$ is an outer measure on $\mathbb{R}^n$, then $\mu$ is Borel if and only if

$$\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2),$$

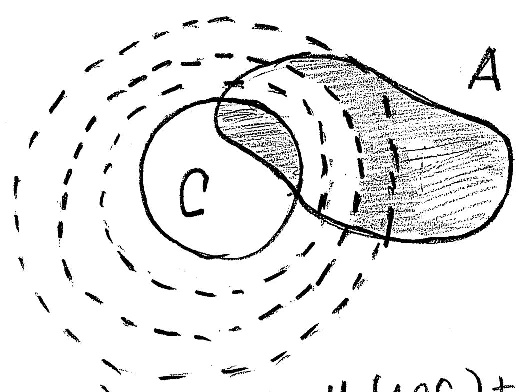
for every $E_1, E_2 \subset \mathbb{R}^n$, $\operatorname{dist}(E_1, E_2) > 0$.

<u>Proof</u>: It suffices to show that: $\forall C$ closed, $\mu(A) \geq \mu(A \cap C) + \mu(A \setminus C)$, $\forall A$.

- $\mu(A) = \infty$ ✓

Assume $\mu(A) < \infty$, $C_k := \{x : d(x, C) \leq \frac{1}{k}\}$

$R_k = (C_k \setminus C_{k+1}) \cap A$




$$d(C \cap A, A \setminus C_k) > 0 \Rightarrow \mu(A \cap C) + \mu(A \setminus C_k) \overset{\text{hypothesis}}{=} \mu[(A \cap C) \cup (A \setminus C_k)] \leq \mu(A)$$

$$\mu(A\cap C)+\mu(A\setminus C)\le \mu(A\cap C)+\mu(A\setminus C_k)+\sum_{j=k+1}^{\infty}\mu(R_j)$$

$$\le \mu(A)+\sum_{j=k+1}^{\infty}\mu(R_j)$$

$$\sum_{s=1}^{N}\mu(R_j)=\sum_{j=1}^{N}\mu(R_{2j})+\sum_{j=1}^{N}\mu(R_{2j-1}), \quad dist\,(R_{2j},R_{2k})>0$$

$$=\mu\Big(\bigcup_{j=1}^{N}R_{2j}\Big)+\mu\Big(\bigcup_{j=1}^{N}R_{2j-1}\Big)$$

$$\le 2\mu(A)<\infty.$$

So $\sum_{j=1}^{\infty}\mu(R_j)<\infty$ and hence $\lim_{k\to\infty}\sum_{j=k}^{\infty}\mu(R_j)=0$. ∎

<u>Ex</u>: $\mathcal{H}^s$ is Borel, $0\le s\le n$ is Borel on $\mathbb{R}^n$

Show first $\mathcal{H}^s_\delta(E_1\cup E_2)=\mathcal{H}^s_\delta(E_1)+\mathcal{H}^s_\delta(E_2)$ if $dist\,(E_1,E_2)>0$. Then, let $\delta\to 0$.

<u>Ex</u>: $\mathcal{L}^n$ is Borel on $\mathbb{R}^n$ (Recall that $\mathcal{L}^n=\mathcal{H}^n$)

<u>Def</u>: $\mu$ is a Borel regular measure if for every $F\subset\mathbb{R}^n$ there exists a Borel set $E$ such that:

$$F\subset E,\quad \mu(E)=\mu(F).$$

<u>Thm</u>: $\mathcal{H}^s$ is Borel regular on $\mathbb{R}^n$. ($\mathcal{L}^n$ is also Borel regular with similar proof).

<u>Proof</u>: Use closed sets in the definition of $\mathcal{H}^s$.

$\forall K \in \mathbb{N}$, $\exists \{F_i^K\}_{i=1}^{\infty}$, $F_i^K$ closed covering of $E$ such that:

$$\operatorname{diam}(F_i^K) \le \frac{1}{K}, \quad \sum_{i=1}^{\infty} \omega_s \left(\frac{\operatorname{diam} F_i^K}{2}\right)^s \le \mathcal{H}^s_{1/K}(E) + \frac{1}{K}$$

$$E \subset \bigcup_{i=1}^{\infty} F_i^K.$$

Let $F = \bigcap_{K=1}^{\infty} \bigcup_{i=1}^{\infty} F_i^K \supseteq E$. Clearly, $\mathcal{H}^s(E) \le \mathcal{H}^s(F)$

We only need $\mathcal{H}^s(F) \le \mathcal{H}^s(E)$:

$$\mathcal{H}^s_{1/K}(F) \le \sum_{i=1}^{\infty} \omega_s \left(\frac{\operatorname{diam} F_i^K}{2}\right)^s \le \mathcal{H}^s_{1/K}(E) + \frac{1}{K}$$

↑ def. of $\mathcal{H}^s_\delta$

Let $K \to \infty$, $\Rightarrow$ $\mathcal{H}^s(F) \le \mathcal{H}^s(E)$. ∎

<u>Ex</u>: Let $\mu = \sum_{i=1}^{\infty} \delta_{1/i}$ on $\mathbb{R}$, $\mu$ is Borel. Let $E = (0,1)$, then $\mu(E) = \infty$. Note that $\mu(E \setminus K) = \infty$, $\forall K \subset E$ compact. Thus, $E$ can not be approximated by compact sets. However we have:

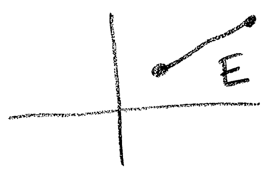
<u>Theorem 1</u>: $\mu$ Borel measure on $\mathbb{R}^n$. $E$ Borel set, $\mu(E) < \infty$. Then:
$\forall \varepsilon > 0$ $\exists K \subset E$, compact $\mu(E \setminus K) < \varepsilon$. In particular:

$$\mu(E) = \sup\{\mu(K) : K \subset E,\ K \text{ compact}\}.$$

The previous example shows that $\mu(E) < \infty$ is needed in Theorem 1.

A locally finite Borel measure $\mu$ on $\mathbb{R}^n$ (i.e. $\mu(K)<\infty$ $\forall K \subset \mathbb{R}^n$ compact) admits outer approximation by open sets.

<u>Ex</u>: Let $\mu = \mathcal{H}^1$ (Borel but not locally finite), measure on $\mathbb{R}^2$.




$\mu(E) = \mathcal{H}^1(E) < \infty$,
but $\mathcal{H}^1(A) = \infty$ $\forall A \subset \mathbb{R}^2$ open

However, we have:

<u>Theorem 2</u>: $\mu$ locally finite Borel measure on $\mathbb{R}^n$, $E$ Borel set. Then

$$\mu(E) = \inf\{\mu(A) : E \subset A,\ A \text{ open}\}$$
$$= \sup\{\mu(K) : K \subset E,\ K \text{ compact}\}.$$

<u>Radon measure</u>: A Radon measure $\mu$ on $\mathbb{R}^n$ is a Borel regular measure such that $\mu(K)<\infty$, $\forall K \subset \mathbb{R}^n$, compact. By Theorem 2:

$$\mu(E) = \inf\{\mu(A) : E \subset A,\ A \text{ open}\}$$
$$= \sup\{\mu(K) : K \subset E,\ K \text{ compact}\},$$

for every Borel set $E$.

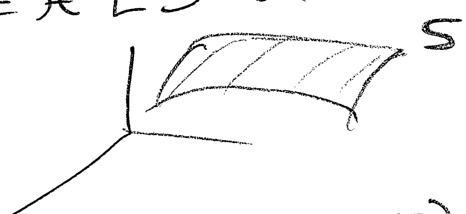
<u>Remark</u>: By Borel regularity, a Radon measure $\mu$ is characterized on $\mathcal{M}(\mu)$ by its value on compact (or open sets).

<u>Ex</u>: Fix $n$, $0 \leq s \leq n$

- $\mathcal{L}^n$ is Radon measure
- $\mathcal{H}^s$ is not Radon measure (Ex. $\mathcal{H}^1([0,1]^2) = \infty$)
- $E$ Borel, $\mathcal{H}^s(E) < \infty$, then

$\mu = \mathcal{H}^s \llcorner E$ is Radon. (if $\mu$ is Borel regular on $\mathbb{R}^n$, $E \in \mathcal{M}(\mu)$, $\mu \llcorner E$ locally finite $\Rightarrow$ $\mu \llcorner E$ is Radon on $\mathbb{R}^n$).

<u>Ex</u>: $\mu = \mathcal{H}^2 \llcorner S$ on $\mathbb{R}^3$ is Radon.




<u>Def</u>: $\mu \llcorner E(F) = \mu(E \cap F)$ restriction of a measure.

By Borel regularity we have:

<u>Theorem 3</u>: $\mu$ Radon measure on $\mathbb{R}^n$:

For <u>every $E \subset \mathbb{R}^n$</u>: $\mu(E) = \inf\{\mu(A) ; E \subset A, A \text{ open}\}$

For <u>every $E \in \mathcal{M}(\mu)$</u>: $\mu(E) = \sup\{\mu(K) : K \subset E, \text{compact}\}$

<u>Remark</u>: $\mu, \nu$ Radon, $\mu(K) = \nu(K)$ $\forall K$ compact $\Rightarrow$ $\mu = \nu$ on $\mathcal{M}(\mu)$.

## Push-forward of a measure:



Let $\mu$ be an outer measure on $\mathbb{R}^n$

Let $f : \mathbb{R}^n \to \mathbb{R}^m$.

The push-forward of $\mu$ through $f$ is the outer measure $f_{\#}\mu$ on $\mathbb{R}^m$ defined by:

$$f_{\#}\mu(E) = \mu(f^{-1}(E)), \quad E \subset \mathbb{R}^m.$$

Ex: $f_{\#}\delta_x = \delta_{f(x)}$

Recall, $\delta_x(E) = \begin{cases} 1, & x \in E \\ 0, & \text{otherwise} \end{cases}$

Prop: $\mu$ Radon, $f : \mathbb{R}^n \to \mathbb{R}^n$ continuous and proper ($f^{-1}$(compact) is compact). Then $f_{\#}\mu$ is Radon, $\operatorname{supp} f_{\#}\mu = f(\operatorname{supp}\mu)$ and $\int_{\mathbb{R}^m} f \, d(f_{\#}\mu) = \int_{\mathbb{R}^n} (u \circ f) \, d\mu$,

$\forall u : \mathbb{R}^n \to [0, +\infty]$ Borel measurable.

Prop: $\mu$ Radon, $E \subset \mathbb{R}^n$ bounded, $\mu(\partial E) = 0$.

Then, $\forall \varepsilon > 0$ $\exists A$ open, $K$ compact such that $\bar{A} \subseteq E \subset \mathring{K}$, $\mu(K \setminus A) < \varepsilon$

Proof: Given $E$, set:

$$A_t = \{x \in \mathring{E} : d(x, \partial E) > t\}$$

$$K_s = \{x \in \mathbb{R}^n : d(x, E) \leq s\}$$

$$\mu(E) = \mathring{E} = \bigcup_{t>0} A_t \Rightarrow \mu(\mathring{E}) = \lim_{t \to 0} \mu(A_t) \quad (1)$$

$$\bar{E} = \bigcap_{s>0} K_s \Rightarrow \mu(\bar{E}) = \lim_{s \to 0} \mu(K_s) \quad (2)$$

From (1), for $t$ small enough, set

$$A := A_t$$

Then

$$\mu(E \setminus A) < \frac{\varepsilon}{2}, \quad A_t \text{ open}, \quad \bar{A}_t \subset E$$

$K_s$ is compact, $E \subset \mathring{K}_s$, if we take $s$ small enough, $\mu(K \setminus E) < \frac{\varepsilon}{2}$

Prop (Foliations by Borel Sets): If $\{E_t\}_{t \in I}$ is a disjoint family of Borel sets in $\mathbb{R}^n$, and $\mu$ is a Radon measure on $\mathbb{R}^n$, then:

$\{t : \mu(E_t) > 0\}$ is at most countable.

Proof: Let $I_k = \{t \in I : \mu(E_t \cap B_k) > \frac{1}{k}\}$

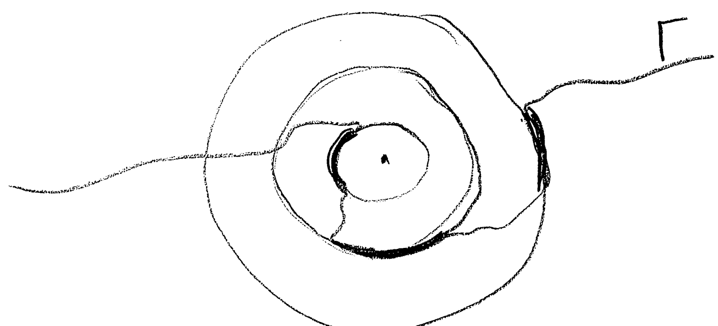
$$\Rightarrow \{t \in I : \mu(E_t) > 0\} = \bigcup_{k=1}^{\infty} I_k$$

$\forall J \subset I_k$ finite,

$$\mu(B_k(0)) \geq \mu\Big(\big(\bigcup_{t\in J} E_t\big) \cap B_k(0)\Big)$$
$$= \sum_{t \in J} \mu(E_t \cap B_k(0))$$
$$\geq \frac{\# J}{k}$$

$\therefore \quad \# I_k \leq k\, \mu(B_k(0)) < \infty \quad \blacksquare$

Ex. As an application of previous Proposition, a curve of locally finite length can contain at most countably many circular arcs of positive length.




$\mathcal{H}^1(\Gamma \cap \partial B(x_0, r)) > 0$ for at most countably many $r > 0$.