Lecture 21

Set operations on Gauss-Green measures.

The following theorem is a rather lengthy (but simple) application of the structure theory of sets of finite perimeter.

Notation: $M_1 \approx M_2 \iff \mathcal{H}^{n-1}(M_1 \Delta M_2) = 0$.

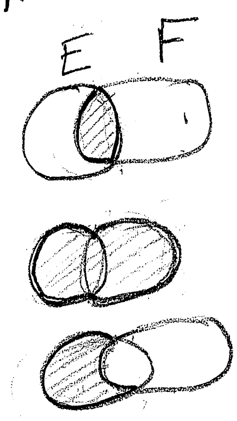
<u>Theorem</u>: E, F sets of locally finite perimeter.

Let

$$\{\nu_E = \nu_F\} = \{x \in \partial^* E \cap \partial^* F : \nu_E(x) = \nu_F(x)\},$$

$$\{\nu_E = -\nu_F\} = \{x \in \partial^* E \cap \partial^* F : \nu_E(x) = -\nu_F(x)\}.$$

then $E \cap F$, $E \setminus F$ and $E \cup F$ are sets of locally finite perimeter, with



$$\mu_{E\cap F} = \mu_E \llcorner F^{(1)} + \mu_F \llcorner E^{(1)} + \nu_E \mathcal{H}^{n-1} \llcorner \{\nu_E = \nu_F\},$$

$$\mu_{E\setminus F} = \mu_E \llcorner F^{(0)} - \mu_F \llcorner E^{(1)} + \nu_E \mathcal{H}^{n-1} \llcorner \{\nu_E = -\nu_F\},$$

$$\mu_{E\cup F} = \mu_E \llcorner F^{(0)} + \mu_F \llcorner E^{(0)} + \nu_E \mathcal{H}^{n-1} \llcorner \{\nu_E = \nu_F\}.$$

and

$$\partial^*(E\cap F) \approx (F^{(1)} \cap \partial^* E) \cup (E^{(1)} \cap \partial^* F) \cup \{\nu_E = \nu_F\}$$

$$\partial^*(E\setminus F) \approx (F^{(0)} \cap \partial^* E) \cup (E^{(1)} \cap \partial^* F) \cup \{\nu_E = -\nu_F\}$$

$$\partial^*(E\cup F) \approx (F^{(0)} \cap \partial^* E) \cup (E^{(0)} \cap \partial^* F) \cup \{\nu_E = \nu_F\},$$

Moreover, for every Borel set $G \subset \mathbb{R}^n$,

$$P(E\cap F; G) = P(E; F^{(1)} \cap G) + P(F; E^{(1)} \cap G) + \mathcal{H}^{n-1}(\{\nu_E = \nu_F\} \cap G).$$

$$P(E\setminus F; G) = P(E; F^{(0)} \cap G) + P(F; E^{(1)} \cap G) + \mathcal{H}^{n-1}(\{\nu_E = -\nu_F\} \cap G).$$

$$P(E\cup F; G) = P(E; F^{(0)} \cap G) + P(F; E^{(0)} \cap G) + \mathcal{H}^{n-1}(\{\nu_E = \nu_F\} \cap G).$$

Remark:

$$\partial^* E \approx E^{(1/2)} \approx \partial^e E$$

$$M \approx (M \cap E^{(1)}) \cup (M \cap E^{(0)}) \cup (M \cap E^{(1/2)}),$$

for every Borel set M and for every set of locally finite perimeter E.

Density estimates for perimeter minimizers.

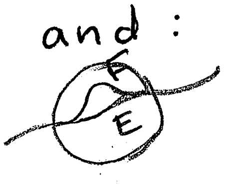
Definition: $A \subset \mathbb{R}^n$ open, bounded. $E \subset \mathbb{R}^n$ a set of locally finite perimeter.
We say that E is a perimeter minimizer in A if $\operatorname{spt} \mu_E = \partial E$ and:

$$P(E; A) \leq P(F; A), \quad \text{whenever } E \Delta F \subset\subset A.$$

Definition: $A \subset \mathbb{R}^n$ open, unbounded (example $A = \mathbb{R}^n$).
$E \subset \mathbb{R}^n$ set of locally finite perimeter.
We say that E is a perimeter minimizer in A if $\operatorname{spt} \mu_E = \partial E$ and:

$$P(E; A') \leq P(F; A'), \quad \forall A' \text{ bounded open } E \Delta F \subset\subset A' \subset A$$

Definition: $A \subset \mathbb{R}^n$ open. We say that E is a local perimeter minimizer in A (at scale r_0) if $\operatorname{spt} \mu_E = \partial E$ and:

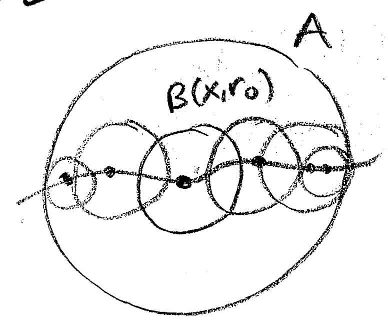


$$P(E; A) \leq P(F; A), \quad \text{whenever } E \Delta F \subset\subset B(x, r_0) \cap A, \; x \in A.$$

In this lecture we will prove the following:

<u>Theorem 1</u> : (Density estimates for local perimeter minimizers). For every $n \geq 2$, there exists $c(n) > 0$ with the following property. If A is an open set in $\mathbb{R}^n$ and E is a local perimeter minimizer in A at scale r_0, then:

For every ball $B(x,r) \subset\subset A$ with $x \in A \cap \partial E$ and $r < r_0$:



$$\frac{1}{2^n} \leq \frac{|E \cap B(x,r)|}{\omega_n r^n} \leq 1 - \frac{1}{2^n}$$

$$c(n) \leq \frac{P(E; B(x,r))}{r^{n-1}} \leq n\,\omega_n$$

In particular,

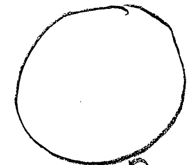
$$\mathcal{H}^{n-1}(A \cap (\partial E \setminus \partial^* E)) = 0.$$

<u>Remark 1</u>: We may take $c(n) = \omega_{n-1}$ (see section 17.4 in textbook). The lower bound ω_{n-1} is sharp, as it is saturated by half-spaces.

<u>Remark 2</u> : The mild regularity $\mathcal{H}^{n-1}((\partial E \setminus \partial^* E) \cap A) = 0$ alone excludes the possibility for perimeter minimizers to present the wild singularities

that generic sets of finite perimeter may show up. Indeed, before proving Theorem 1, we present a "wild" set of finite perimeter that can not be a minimizer. If $n \geq 2$, given $\varepsilon > 0$ we now constuct a set of finite perimeter $E \subset B$ s.t:

$$|E| \leq \varepsilon \, , \qquad |\operatorname{spt} \mu_E| \geq \omega_n - \varepsilon \, .$$



$B = B(0,1)$

Since:

$$\operatorname{spt} \mu_E \subset \partial E$$

then:

$$|\partial E| \geq |\operatorname{spt} \mu_E| \geq \omega_n - \varepsilon > 0$$

$$\therefore \quad |\partial E| > 0$$

$$\therefore \quad \mathcal{H}^{n-1}(\partial E) = \infty$$

Morever, if $|E \Delta F| = 0$ then, since $\mu_F = \mu_E$, we have:

$$|\partial F| > 0 \Rightarrow \mathcal{H}^{n-1}(\partial F) = \infty \, .$$

To constuct such E, let $\{x_i\}_{i=1}^{\infty}$ be a dense set in $B = B(0,1)$ and $\{r_i\}$, $r_i < \varepsilon$, such that:

$$n \omega_n \sum_{i=1}^{\infty} r_i^{n-1} \leq 1 \, .$$

Define:

$$E := \bigcup_{i=1}^{\infty} B_i \, , \quad B_i = B(x_i, r_i) \subset B$$

We have:

$$P(B_i) = \mathcal{H}^{n-1}(\partial B_i) = n\omega_n r_i^{n-1}$$

$$E_N := \bigcup_{i=1}^{N} B_i$$

$$P(E_N) \leq \sum_{i=1}^{N} P(B_i) \leq n\omega_n \sum_{i=1}^{\infty} r_i^{n-1} \leq 1$$

$E_N \to E$ in $L^1(B) \Rightarrow$

$$P(E) \leq \liminf_{N\to\infty} P(E_N)$$
$$\leq 1$$

$\therefore$ <u>E is a set of finite perimeter.</u>

Now;

$$|E| \leq \omega_n \sum_{i=1}^{\infty} r_i^n \leq \varepsilon n\omega_n \sum_{i=1}^{\infty} r_i^{n-1} \quad ; \text{ since } r_i < \varepsilon$$
$$\leq \varepsilon.$$

$\therefore$ <u>$|E| \leq \varepsilon$</u>, and hence <u>$|B\setminus E| \geq \omega_n - \varepsilon$</u>

<u>Claim</u>: $|\operatorname{spt}\mu_E| = |B\setminus E|$

Let $x \in \operatorname{spt}\mu_E$. Since E is open and $\operatorname{spt}\mu_E \subset \partial E$, then $x \in \bar{B}\setminus E$. Hence $\operatorname{spt}\mu_E \subset \bar{B}\setminus E$. Now, since $\{x_i\}$ is dense in B, $\forall x \in B$ and $\forall r>0$, we have $|E\cap B(x,r)|>0$. Also, for $\mathcal{L}^n$-a.e. $x\in\mathbb{R}^n\setminus E$, by the Corollary in Lecture 7 (Page 7.4) we have:

$$\frac{|E\cap B(x,r)|}{|B(x,r)|} \to 0, \text{ as } r\to 0$$

Therefore, for $\mathcal{L}^n$-a.e. $x\in B\setminus E$, $0<|E\cap B(x,r)|<\omega_n r^n$ $\forall r>0$.

Since $\operatorname{spt}\mu_E = \{x\in\mathbb{R}^n : 0<|E\cap B(x,r)|<\omega_n r^n, \forall r>0\}$, the claim follows.

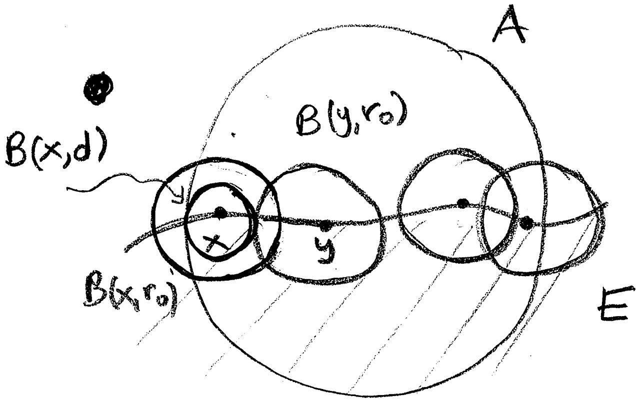
Proof of Theorem 1 :

- Since we are assuming $\operatorname{spt} \mu_E = \partial E$ we have:

$$0 < |E \cap B(x,r)| < \omega_n r^n, \quad \forall x \in \partial E, \ r > 0$$

- Let $x \in A \cap \partial E$. Let

$$d = \min\{\operatorname{dist}(x, \partial A), r_0\}$$



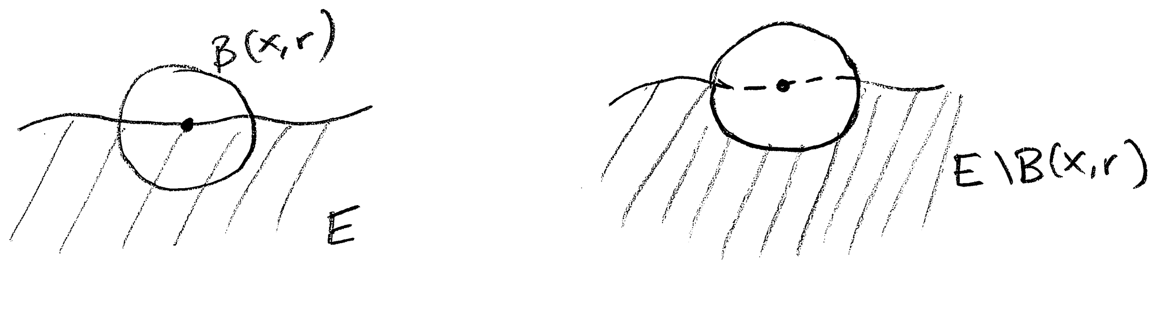
Since $|\mu_E| = \mathcal{H}^{n-1} \llcorner \partial^* E$ is a Radon measure then by our Proposition on "foliations by Borel sets" in Lecture 2 (Page 2.7), we have:

$$|\mu_E|(\partial B(x,r)) = 0 \quad \text{for a.e. } r < d$$

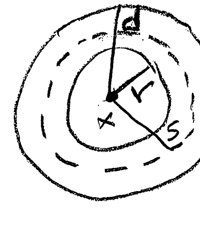
$$\therefore \quad \boxed{\mathcal{H}^{n-1}(\partial^* E \cap \partial B(x,r)) = 0 \text{ for a.e. } r < d} \quad (1)$$

- Let:

$$F = E \setminus B(x,r)), \quad \text{a.e. } r < d$$



Since $E \Delta F \subset\subset B(x,s) \subset A$ for any $s \in (r,d)$, then by local minimality:

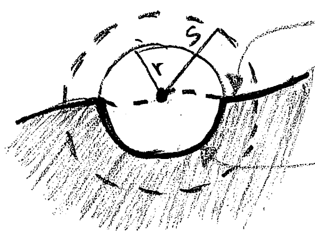


$$P(E; B(x,s)) \leq P(F; B(x,s))$$

$$= P(E \setminus B(x,r); B(x,s)) \quad , \text{ a.e. } r < d$$

From the theorem on set operations on Gauss-Green measures (Page 21.1) we have, using (1), that:

$$P(E \setminus B(x,r); B(x,s)) = \mathcal{H}^{n-1}(E^{(1)} \cap \partial B(x,r)) + P(E; B(x,s) \setminus \overline{B(x,r)})$$



Hence; for any $s \in (r,d)$

$$P(E; B(x,s)) \leq \mathcal{H}^{n-1}(E^{(1)} \cap \partial B(x,r)) + P(E; B(x,s) \setminus \overline{B(x,r)}) \tag{2}$$

Letting $s \to r^+$ in (2) gives:

$$\boxed{P(E; B(x,r)) \leq \mathcal{H}^{n-1}(E^{(1)} \cap \partial B(x,r)) \quad (3)}$$

$\forall x \in A \cap \partial E$ and a.e. $r < d$

Since $\mathcal{H}^{n-1}(E^{(1)} \cap \partial B(x,r)) \leq P(B(x,r)) = n\omega_n r^{n-1}$

$$\therefore \quad P(E; B(x,r)) \leq n\omega_n r^{n-1} \quad \text{ a.e. } r < d$$

Since $r \mapsto P(E; B(x,r))$ is increasing we obtain

$$\boxed{P(E; B(x,r)) \leq n\omega_n r^{n-1}, \ \forall r < d, \ \forall x \in A \cap \partial E}$$

and we have proved the upper bound

$$\boxed{\frac{P(E; B(x,r))}{r^{n-1}} \leq n\,\omega_n \quad \forall x \in \partial E \cap A, \ \forall r < d}$$

in Theorem 1.

Adding $\mathcal{H}^{n-1}(E^{(1)} \cap \partial B(x,r))$ to both sides of (3):

$$P(E; B(x,r)) + \mathcal{H}^{n-1}(E^{(1)} \cap \partial B(x,r)) \leq \mathcal{H}^{n-1}(E^{(1)} \cap \partial B(x,r)) + \mathcal{H}^{n-1}(E^{(1)} \cap \partial B(x,r)),$$

and using (1) and the theorem on set operations for Gauss-Green measures yields:

$$P(E \cap B(x,r)) \leq 2\mathcal{H}^{n-1}(E^{(1)} \cap \partial B(x,r)) \quad \forall x \in \partial E \cap A \text{ a.e. } r < d$$

By the Euclidean isoperimetric inequality:

$$n\omega_n^{1/n} |B(x,r) \cap E|^{\frac{n-1}{n}} \leq P(E \cap B(x,r)) \leq 2\mathcal{H}^{n-1}(E^{(1)} \cap \partial B(x,r))$$

We can now proceed as in Lecture 19:

$m: (0,\infty) \to [0,\infty)$, $m(s) = |E^{(1)} \cap B(x,s)| = |E \cap B(x,s)|$, $s > 0$ is absolutely continuous,

$$m'(s) = \mathcal{H}^{n-1}(E^{(1)} \cap \partial B(x,s)) \quad \text{a.e. } s > 0.$$

Therefore:

$$n\omega_n^{1/n} m(r)^{\frac{n-1}{n}} \leq 2m'(r), \quad \text{a.e. } r < d.$$

$m(r) > 0$ since $x \in \partial E = \operatorname{spt} \mu_E$. Thus:

$$n\omega_n^{1/n} \leq 2\, m(r)^{-1+\frac{1}{n}}\, m'(r)$$

$$n\omega_n^{1/n} \leq 2\, \frac{d}{dr}\left[n\, m(r)^{1/n}\right]$$

$$\therefore\ \cancel{n}\, m(r)^{1/n} \geq \frac{\cancel{n}\,\omega_n^{1/n}}{2}\, r$$

$$m(r) \geq \frac{\omega_n r^n}{2^n}$$

$$\boxed{\frac{|E \cap B(x,r)|}{\omega_n r^n} \geq \frac{1}{2^n}, \quad \forall r<d,\ \forall x \in \partial E \cap A}$$

We can repeat the above argument for $\mathbb{R}^n \setminus E$, since $\mathbb{R}^n \setminus E$ is a local perimeter minimizer in A (with constant r_0) and $x \in \partial E = \operatorname{spt} \mu_E$ implies $|(\mathbb{R}^n \setminus E) \cap B(x,s)| > 0$ $\forall s>0$. Therefore:

$$\frac{|(\mathbb{R}^n \setminus E) \cap B(x,r)|}{\omega_n r^n} \geq \frac{1}{2^n}, \quad \forall r<d,\ \forall x \in \partial E \cap A;$$

that is,

$$\boxed{\frac{|E \cap B(x,r)|}{\omega_n r^n} \leq 1 - \frac{1}{2^n}, \quad \forall r<d,\ \forall x \in \partial E \cap A}$$

Finally, by the relative isoperimetric inequality:

$$P(E; B(x,r)) \geq \tilde{c}(n)\, |E \cap B(x,r)|^{\frac{n-1}{n}} \geq \frac{\tilde{c}(n)\, \omega_n^{\frac{n-1}{n}}}{2^{n-1}}\, r^{n-1} = c(n)\, r^{n-1}$$

$$\therefore\ \boxed{\frac{P(E; B(x,r))}{r^{n-1}} \geq c(n), \quad \forall r<d,\ \forall x \in A \cap \partial E}$$

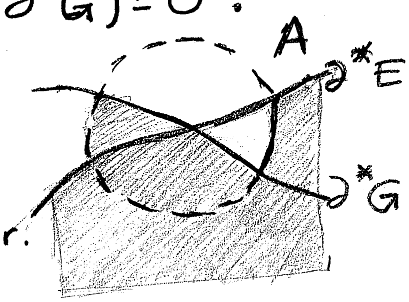
Also, notice that $A \cap \partial E \subset A \cap \partial^e E \implies \mathcal{H}^{n-1}(A \cap (\partial E \setminus \partial^* E)) = 0$ by Federer's theorem. ∎

Comparison sets by replacement.

The following theorem will be used later, the technical proof does not introduce new ideas (see proof in textbook, chapter 16).

Theorem ; $E, G \subset \mathbb{R}^n$ sets of locally finite perimeter. A open set of finite perimeter such that:

$$\mathcal{H}^{n-1}(\partial^* A \cap \partial^* E) = \mathcal{H}^{n-1}(\partial^* A \cap \partial^* G) = 0.$$



Define:

$$F = (G \cap A) \cup (E \setminus A)$$

Then, F is a set of finite perimeter. Moreover, if $A \subset\subset A'$, A' open, then

$$P(F; A') = P(G; A) + P(E; A' \setminus \bar{A}) + \mathcal{H}^{n-1}((E^{(1)} \Delta G^{(1)}) \cap \partial^* A).$$

Corollary : If A, E and G are as in previous theorem. If E is a perimeter minimizer in some open set A', $A \subset\subset A'$, then:

$$P(E; A) \leq P(G; A) + \mathcal{H}^{n-1}((E^{(1)} \Delta G^{(1)}) \cap \partial^* A).$$

Moreover, if $E \Delta G \subset\subset A$, then:

$$P(E; A) \leq P(G; A)$$