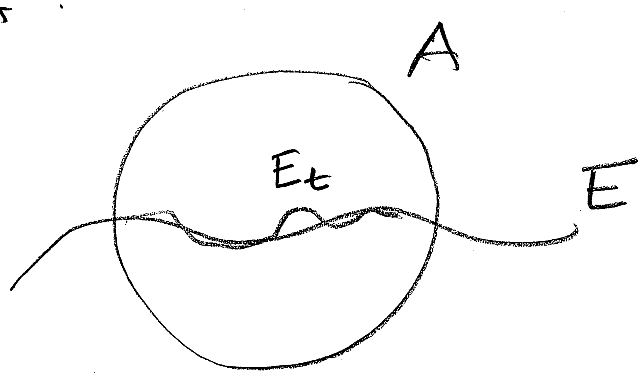
Lesson 22

First Variation of Perimeter

Let E be a perimeter minimizer in A.
Let E_t be a small perturbation of E inside A.



Let $T \in C_c^\infty(A; \mathbb{R}^n)$,
$f_t(x) := x + tT(x)$
If $|t| << 1$ thet $f_t \in C^\infty$ is a diffeomorphism.

Let:

$$E_t := f_t(E),$$

hence:

$$E_t \Delta E \subset\subset A, \quad \text{for } |t| << 1$$

At least formally, if E is a minimizer in A:

$$\frac{d}{dt} P(E_t; A) = 0$$

$$\frac{d^2}{dt^2} P(E_t; A \Big|_{t=0} \geq 0.$$

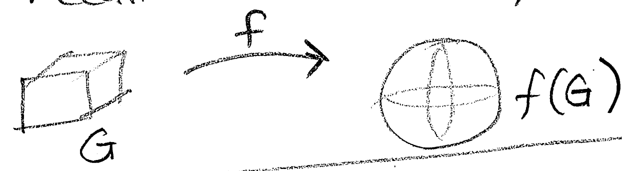
<u>Lemma</u>: E set of locally finite perimeter, $f: \mathbb{R}^n \to \mathbb{R}^n$ diffeomorphism, $g := f^{-1}$. Then $f(E)$ has locally finite perimeter and:

$$\int_{\partial^*(f(E))\cap F} \nu_{f(E)}\, d\mathcal{H}^{n-1} = \int_{\partial^* E\cap g(F)} (\nabla g)^*\circ f\ \nu_E\, Jf\, d\mathcal{H}^{n-1}, \quad \forall F\subset\mathbb{R}^n \text{ Borel}$$

Proof:

- <u>Claim</u>: $u_i \to u$ in $L^1_{loc}(\mathbb{R}^n)$ then $u_i\circ g \to u\circ g$ in $L^1_{loc}(\mathbb{R}^n)$

Indeed, recall the area formula:



$$\boxed{\int_{f(G)} h(y)\,dy = \int_G h(f(x))\,Jf(x)\,dx} \quad (1)$$

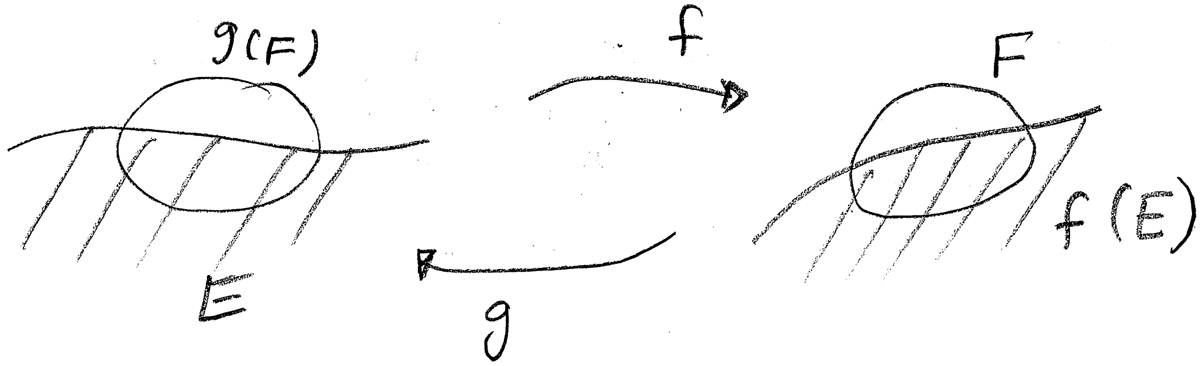
Hence, if $K\subset\mathbb{R}^n$ is compact, then $g(K)$ is compact, and using (1) (with $G=g(K)$, $h=|u_i\circ g-u\circ g|$):

$$\int_{f(g(K))} |u_i\circ g - u\circ g| = \int_{g(K)} |(u_i\circ g - u\circ g)\circ f|\ Jf(x)$$

$$\therefore \int_K |u_i\circ g - u\circ g| = \int_{g(K)} |u_i - u|\, Jf$$

$$\leq \mathrm{Lip}(f;g(K))^n \int_{g(K)} |u_i - u|; \quad Jf \leq \mathrm{Lip}(f;g(K))^n$$

$$\to 0$$

- 

Let:

$u = \chi_E$, $\quad u_\varepsilon = (\chi_E) * \rho_\varepsilon$

$v = \chi_{f(E)} = \chi_E \circ g \quad (x \in f(E) \Leftrightarrow x = f(z),\ z \in E$
$\Leftrightarrow g(x) = z$
$\Leftrightarrow g(x) \in E)$

$v_\varepsilon = u_\varepsilon \circ g$

Let:
$T \in C_c^1(\mathbb{R}^n)$
$S = T \circ g \in C_c^1(\mathbb{R}^n)$

Now

$$\int_{f(E)} \operatorname{div} S = \int_{\mathbb{R}^n} \chi_{f(E)} \operatorname{div} S$$

$$= \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} v_\varepsilon \operatorname{div} S \text{ ; by claim } v_\varepsilon = u_\varepsilon \circ g \xrightarrow{L^1_{loc}} u \circ g = \chi_{f(E)}$$

$$= -\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} S \cdot \nabla v_\varepsilon \quad , \quad v_\varepsilon = u_\varepsilon \circ g$$

$$= -\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} S \cdot \left[(\nabla u_\varepsilon \circ g) \nabla g \right] \text{ ; Chain rule } \Rightarrow \nabla v_\varepsilon(x) = \nabla u_\varepsilon(g(x)) \nabla g(x)$$

$$= -\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} S \cdot \left[(\nabla g)^* \, \nabla u_\varepsilon \circ g \right]$$

$$= -\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} (S \circ f) \cdot \left[(\nabla g)^* \circ f \ \nabla u_\varepsilon \right] Jf \text{ ; by Area Formula}$$

$$= -\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} T \cdot \left[(\nabla g)^* \circ f \ \nabla u_\varepsilon \right] Jf .$$

Using that:

$$u_\varepsilon \to \chi_E \text{ in } L^1_{loc}(\mathbb{R}^n$$

$$-\nabla u_\varepsilon \xrightarrow{*} \mu_E = \nu_E \mathcal{H}^{n-1} \llcorner \partial^* E$$

yields:

$$\int_{f(E)} \operatorname{div} S = \int_{\partial^* E} T \cdot [(\nabla g)^* \circ f \, \nu_E] \, Jf \, d\mathcal{H}^{n-1},$$

for every $S, T \in C^1_c(\mathbb{R}^n; \mathbb{R}^n)$.

Taking the sup over all S and T, $|S| \le 1$, $|T| \le 1$

$\Rightarrow$ $f(E)$ is a set of locally finite perimeter

and hence:

$$\boxed{\int_{\partial^* f(E)} S \cdot \nu_{f(E)} \, d\mathcal{H}^{n-1} = \int_{\partial^* E} T \cdot [(\nabla g)^* \circ f \, \nu_E] \, Jf \, d\mathcal{H}^{n-1} \qquad (2)}$$

Take $S = (\varphi e) * \rho_\varepsilon$ in (2) for $\varphi \in C_c(\mathbb{R}^n)$ and $e \in S^{n-1}$; letting $\varepsilon \to 0^+$:

$$e \cdot \int_{\partial^* f(E)} \varphi \, \nu_{f(E)} \, d\mathcal{H}^{n-1} = e \cdot \int_{\partial^* E} (\varphi \circ f) [Jf (\nabla g)^* \circ f] \, \nu_E \, d\mathcal{H}^{n-1}$$

$\forall e \in S^{n-1}$. Hence:

$$\boxed{\int_{\partial^* f(E)} \varphi \, \nu_{f(E)} \, d\mathcal{H}^{n-1} = \int_{\partial^* E} (\varphi \circ f) [Jf \, (\nabla g)^* \circ f] \, \nu_E \, d\mathcal{H}^{n-1} \qquad (3)}$$

$\forall \varphi \in C_c(\mathbb{R}^n)$.

Let $F \subset \mathbb{R}^n$ be a Borel set.

By (3) and an approximation argument:

$$\int_{F \cap \partial^* f(E)} \nu_{f(E)} \, d\mathcal{H}^{n-1} = \int_{g(F) \cap \partial^* E} \left[Jf \, (\nabla g)^* \circ f \right] \nu_E \, d\mathcal{H}^{n-1}, \quad (4)$$

which proves the Lemma. ∎

<u>Remark</u>: Taking total variations in (4) gives:

$$\mathcal{H}^{n-1}(F \cap \partial^* f(E)) = \int_{g(F) \cap \partial^* E} Jf \, |(\nabla g)^* \circ f \, \nu_E| \, d\mathcal{H}^{n-1}$$

<u>Note</u>: We are going to apply this Remark later and:

f will be $f_t = x + tT(x) \Rightarrow \nabla f_t = I + t\nabla T$.

We need to compute:

Jf_t and $(\nabla g_t)^* \circ f_t$

Lemma (Taylor Expansion of det):

Let $Z \in M_{n\times n}(\mathbb{R})$. Then:

(a) $(I+tZ)^{-1} = I - tZ + t^2Z^2 + O(t^3)$

(b) $\det(I+tZ) = 1 + tr(Z)\cdot t + \frac{t^2}{2}\left(tr(Z)^2 - tr(Z^2)\right) + O(t^3)$

Proof:

(a) For t small enough, $I+tZ$ is invertible.

For (a) use Neumann Series for $(I+tZ)^{-1}$, with $|tZ|<1$.

(b) Case 1: Z is symmetric. In this case, there exists and orthonormal basis $\{v_i\}_{i=1}^n$ of $\mathbb{R}^n$ such that:

$$Z = \sum_{i=1}^n \lambda_i\, v_i \otimes v_i$$

Here, we are using that Z is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$ (i.e., $Z \in \mathbb{R}^n \otimes \mathbb{R}^n$) and hence can be written as a sum of linear maps of the form:

$$(w\otimes v)(x) = (v\cdot x)\, w, \quad \forall x \in \mathbb{R}^n, \quad v, w \in \mathbb{R}^n.$$

In the general case, if $T \in \mathbb{R}^m \otimes \mathbb{R}^n$, and $V = \{v_j\}_{j=1}^n$, $W = \{w_i\}_{i=1}^m$ are orthonormal basis of $\mathbb{R}^n$ and $\mathbb{R}^m$, then:

$$T = \sum_{j=1}^n \sum_{i=1}^m \left(w_i \cdot (Tv_j)\right) w_i \otimes v_j$$

Now:

$$\det(I+tZ) = \prod_{i=1}^{n} (1+t\lambda_i) \quad ; \quad Z \rightsquigarrow \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

$$= 1 + t\sum_{i=1}^{n} \lambda_i + t^2 \sum_{1\le i,j\le n} \lambda_i \lambda_j + O(t^3)$$

$$= 1 + t\,tr(Z) + \frac{t^2}{2}\left(\left(\sum_{i=1}^{n}\lambda_i\right)^2 - \sum_{i=1}^{n} \lambda_i^2\right) + O(t^3)$$

$$= 1 + t\,tr(Z) + \frac{t^2}{2}\left(tr(Z)^2 - tr(Z^2)\right) + O(t^3)$$

Case 2: We decompose:

$$Z = X + Y, \quad X = \frac{Z+Z^*}{2}, \quad Y = \frac{Z-Z^*}{2}$$

Z^* adjoint operator. If Z is identified with a matrix M in some orthonormal base, then: $Z^* = M^T$.

X is symmetric

Y is antisymmetric.

Define:

$\Phi: \mathbb{R}^2 \to \mathbb{R}$ as $\Phi(s,t) = \det(Id + tX + sY)$, $s,t\in\mathbb{R}$

Claim: $\frac{\partial\Phi}{\partial t}(0,0) = tr(Z)$, $\frac{\partial^2\Phi}{\partial t^2}(0,0) = tr(Z)^2 - tr(X^2)$

$\frac{\partial\Phi}{\partial s}(0,0) = \frac{\partial^2\Phi}{\partial s\partial t}(0,0) = 0$, $\frac{\partial^2\Phi}{\partial s^2}(0,0) = -tr(Y^2)$

Indeed, by (b) Case 1:

$$\frac{\partial\Phi}{\partial t}(0,0) = tr(X) = tr(Z) \qquad \frac{\partial^2\Phi}{\partial t^2}(0,0) = tr(X)^2 - tr(X^2)$$
$$= tr(Z)^2 - tr(X^2)$$

Note:

$$\Phi(s,t) = \det(I + tX + sY)$$
$$= \det(S_t + sY) \quad ; \quad S_t = I + tX \text{, symmetric}$$
$$= \det((S_t + sY)^*)$$
$$= \det(S_t + sY^*)$$
$$= \det(S_t - sY) \quad ; \quad \text{since } Y^* = -Y$$
$$= \Phi(-s,t)$$

Thus, Φ is even in $s \Rightarrow$

$$\boxed{\frac{\partial \Phi}{\partial s}(0,0) = 0, \quad \frac{\partial^2 \Phi}{\partial s \partial t}(0,0) = 0}$$

$$\Phi(s,0)^2 = \Phi(s,0)\Phi(-s,0) = \det(I + sY)\det(I - sY)$$
$$= \det(I - s^2 Y^2)$$

Since Y^2 is symmetric, by (b) Step 1 we have:

$$\Phi(s,0)^2 = 1 - s^2 \operatorname{tr}(Y^2) + O(s^4) \longrightarrow (5)$$

$$\Rightarrow \frac{\partial^2}{\partial s^2}(\Phi(s,0)^2)\Big|_{s=0} = 2\,\Phi(0,0)\frac{\partial^2 \Phi}{\partial s^2}(0,0) + 2\left(\frac{\partial \Phi}{\partial s}(0,0)\right)^2$$
$$= 2 \cdot 1 \; \frac{\partial^2 \Phi}{\partial s^2}(0,0) + 2 \cdot 0$$

and by (5):

$$2\,\frac{\partial^2 \Phi}{\partial s^2}(0,0) = -2\operatorname{tr}(Y^2) \Rightarrow \boxed{\frac{\partial^2 \Phi}{\partial s^2}(0,0) = -\operatorname{tr}(Y^2)}$$

Finally, the second order approximation for the function of two variables $\Phi(s,t)$ is:

$$\Phi(s,t) = \Phi(0,0) + \frac{\partial \Phi}{\partial s}(0,0)s + \frac{\partial \Phi}{\partial t}(0,0)t$$
$$+ \frac{1}{2}\left(\frac{\partial^2 \Phi}{\partial s^2}(0,0)s^2 + \frac{\partial^2 \Phi}{\partial s \partial t}(0,0)st + \frac{\partial^2 \Phi}{\partial t^2}(0,0)t^2\right)$$
$$+ R_2(s,t),$$

with $\dfrac{R_2(s,t)}{|(s,t)|^2} \to 0$ as $(s,t) \to 0$.

Using our claim we have:

$$\Phi(t,t) = 1 + 0\cdot s + t\,tr(Z) + \frac{1}{2}\left(-t^2\, tr(Y^2) + 0 + t^2(tr(Z)^2 - tr(X^2))\right) + O(t^3)$$

$$= 1 + t\, tr(Z) + \frac{t^2}{2}\left(tr(Z)^2 - tr(X^2) - tr(Y^2)\right) + O(t^3)$$

Now,

$$tr(Z^2) = tr[(X+Y)(X+Y)]$$
$$= tr[X^2 + Y^2 + XY + YX]$$
$$= tr(X^2) + tr(Y^2) + \underbrace{tr(XY+YX)}_{=0}$$

because $XY+YX$ is antisymmetric:

$$(XY+YX)^* = (Y^*X^* + X^*Y^*) = -YX - XY = -(XY+YX).$$

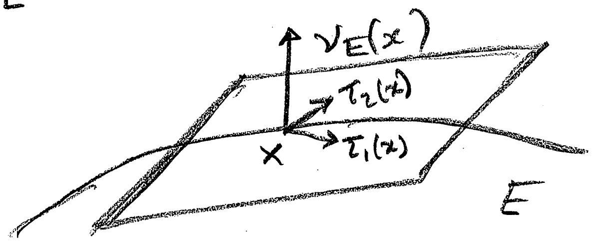
We conclude:

$$\det(I+tZ) = 1 + t\, tr(Z) + \frac{t^2}{2}\left(tr(Z)^2 - tr(Z^2)\right) + O(t^3). \quad \blacksquare$$

22.10

<u>Def.</u>: If E has locally finite perimeter and $T \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$ then the <u>tangential divergence of T</u> is:

$$\operatorname{div}_E(T)(x) = \operatorname{div} T(x) - \nu_E(x)\cdot\left(\nabla T(x)\,\nu_E(x)\right)$$



Let $\{\tau_1(x), \ldots, \tau_{n-1}(x)\}$ be an orthonormal basis of $\nu_E(x)^\perp$. We have:

$$\operatorname{div} T(x) = \sum_{i=1}^{n-1} \tau_i(x)\cdot\left(\nabla T(x)\,\tau_i(x)\right) + \nu_E(x)\cdot\left(\nabla T(x)\,\nu_E(x)\right)$$

$$\operatorname{div} T(x) = \operatorname{tr} \nabla T(x)$$

<u>Note</u>: $\operatorname{div}_E(T)$ only depends on the behavior of T on ∂E because the ν_E direction has been removed.

<u>Def</u>: $\{f_t\}_{|t|<\varepsilon}$ is a <u>local variation in the open set A</u> if:

- $f_0(x) = x \quad \forall x \in \mathbb{R}^n$
- $\{x\in\mathbb{R}^n : f_t(x) \neq x\} \subset\subset A \quad \forall |t|<\varepsilon$.

The initial velocity of $\{f_t\}_{|t|<\varepsilon}$

is
$$T(x) := \frac{\partial f_t}{\partial t}(x,0), \quad x \in \mathbb{R}^n$$

We have:
$$f_t(x) = x + t\,T(x) + O(t^2)$$
$$\nabla f_t(x) = Id + t\,\nabla T(x) + O(t^2)$$

Conversely, if $T \in C_c^\infty(A, \mathbb{R}^n)$ we can construct a local variation $\{f_t\}_{|t|<\varepsilon}$ in A having T as its initial velocity as follows;
$$f_t(x) = x + t\,T(x), \quad x \in \mathbb{R}^n,$$

We now compute the first variation of perimeter (relative to A) with respect to local variations $\{f_t\}_{|t|<\varepsilon}$ in A. That is, we want to compute:
$$\frac{d}{dt}\Big|_{t=0} P(f_t(E); A), \quad \text{for } T \in C_c^\infty(A; \mathbb{R}^n) \text{ given}$$

Theorem 1 (First variation of perimeter): $A \subset \mathbb{R}^n$ open, E set of locally finite perimeter, $\{f_t\}_{|t|<\varepsilon}$ is a local variation in A. Then:
$$P(f_t(E); A) = P(E; A) + t\int_{\partial^* E} \operatorname{div}_E T \, d\mathcal{H}^{n-1} + O(t^2),$$
where T is the initial velocity of $\{f_t\}_{|t|<\varepsilon}$.

Proof: By the Lemma proved at the beginning of this Lecture (see Remark in Page 22.5):

$$P(f_t(E);A)=\int_{A\cap\partial^*E} Jf_t\,|(\nabla g_t)^*\circ f_t\,\nu_E|\,d\mathcal{H}^{n-1}$$

$g_t=f_t^{-1}$, We have:

$\nabla f_t = Id + t\nabla T + O(t^2)$.

Using our Lemma on the Taylor expansion for det we have:

$$\begin{aligned}\nabla g_t\circ f_t &= (\nabla f_t)^{-1}; \quad \text{since } g_t(f_t(x))=x \Rightarrow \nabla g_t(f_t(x))\cdot\nabla f_t(x)=Id\\ &=(Id+t\nabla T+O(t^2))^{-1}\\ &=\boxed{Id-t\nabla T+O(t^2)}\end{aligned}$$

$$\begin{aligned}Jf_t &= |\det \nabla f_t|\\ &=\det(Id+t\nabla T+O(t^2))\\ &=1+t\,tr(\nabla T)+O(t^2)\\ &=\boxed{1+t\,div\,T+O(t^2)}; \quad tr(\nabla T)=div\,T\end{aligned}$$

Now:

$$\begin{aligned}|(\nabla g_t)^*\circ f_t\,\nu_E|^2 &= |\nu_E-t(\nabla T)^*\nu_E|^2+O(t^2)=1-2t\nu_E\cdot((\nabla T)^*\nu_E)+O(t^2)\\ &=1-2t\nu_E\cdot(\nabla T\nu_E)+O(t^2); \quad \text{since } (Sx)\cdot y = x\cdot(S^*y),\ S:\mathbb{R}^n\to\mathbb{R}^n \text{ linear.}\end{aligned}$$

$$\Rightarrow |(\nabla g_t)^*\circ f_t\,\nu_E| = \sqrt{1-2t\nu_E\cdot(\nabla T\nu_E)+O(t^2)} = \boxed{1-t\nu_E\cdot(\nabla T\nu_E)+O(t^2)}$$

$$\begin{aligned}\Rightarrow P(f_t(E);A) &= \int_{A\cap\partial^*E}(1+t\,div\,T+O(t^2))(1-t\nu_E\cdot(\nabla T\nu_E)+O(t^2))\,d\mathcal{H}^{n-1}\\ &=\int_{A\cap\partial^*E} 1+t(div_E T-\nu_E\cdot(\nabla T\nu_E))\,d\mathcal{H}^{n-1}+O(t^2)=\int_{A\cap\partial^*E}(1+t\,div_E T)\,d\mathcal{H}^{n-1}+O(t^2).\end{aligned}$$

$\blacksquare$