Lecture 23

<u>Def</u>: We say that a set of locally finite perimeter E is <u>stationary for perimeter</u> in an open set A if $\operatorname{spt}\mu_E = \partial E$, and

$$\frac{d}{dt}\Big|_{t=0} P(f_t(E); A) = 0,$$

whenever $\{f_t\}_{|t|<\varepsilon}$ is a local variation in A.

<u>Remark 1</u>: E minimizer in $A \Rightarrow E$ is stationary in A.

Indeed, E minimizer in $A \Rightarrow$

$$P(E;A) \leq P(f_t(E);A) \longrightarrow (1)$$

Also,

$$\frac{P(f_t(E);A) - P(E;A)}{t} = \int_{\partial^* E} \operatorname{div}_E T + \frac{O(t^2)}{t}; \quad f_t(x) = x + tT(x) + O(t^2)$$

See Theorem 1, Lecture 22.

$$\therefore \lim_{t\to 0^+} \frac{P(f_t(E);A) - P(E;A)}{t} = \int_{\partial^* E} \operatorname{div}_E T \geq 0; \text{ by } (1)$$

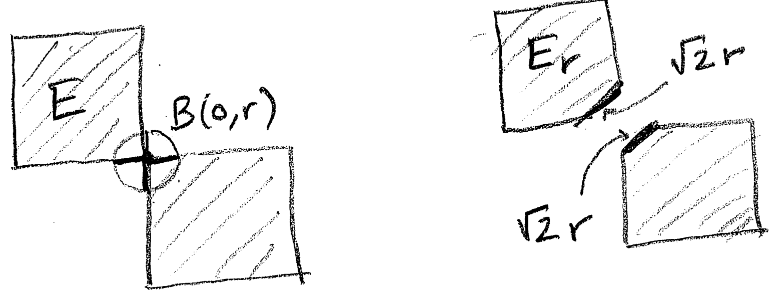
And:

$$\lim_{t\to 0^-} \frac{P(f_t(E);A) - P(E;A)}{t} = \int_{\partial^* E} \operatorname{div}_E T \leq 0; \text{ by } (1)$$

$$\therefore \int_{\partial^* E} \operatorname{div}_E T = 0 \Rightarrow E \text{ is stationary.}$$

Remark 2: The converse is not true.

Consider $E = \{x \in \mathbb{R}^2 : x_1 x_2 < 0\}$:



Note: $\sqrt{2}r + \sqrt{2}r < 4r$

$\Rightarrow$ E is not a minimizer

However, E is stationary.

A fundamental theorem in the analysis of singularities for minimizers is the monotonicity formula:

Theorem (Monotonicity formula). If E is stationary for the perimeter in A and $x_0 \in A \cap \partial E$; then for a.e. $r \in (0, \operatorname{dist}(x_0, \partial A))$,

$$\frac{d}{dr} \frac{P(E; B(x_0,r))}{r^{n-1}} = \frac{d}{dr} \int_{B(x_0,r) \cap \partial^* E} \frac{(\nu_E(x) \cdot (x - x_0))^2}{|x - x_0|^{n+1}} \, d\mathcal{H}^{n-1}$$

We will prove the monotonicity formula later. In this lecture, we will prove a simpler version, stating only the monotonicity of density ratios. Then, as a corollary, we will show that $\mathcal{H}^{n-1}(A \cap (\partial E \setminus \partial^* E)) = 0$, for any stationary set E in A, and the existence of $(n-1)$-dimensional densities everywhere on the topological boundary.

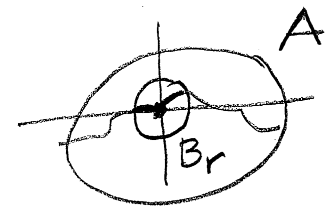
We have:

Theorem 1 (Monotonicity of density ratios). If E is stationary for perimeter in the open set A and $x_0 \in A$, then the density ratios:

$$\frac{P(E; B(x_0,r))}{\omega_{n-1} r^{n-1}}$$

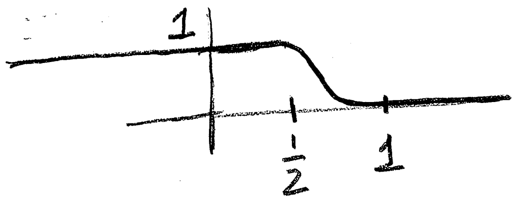
are increasing on $r \in (0, \operatorname{dist}(x_0, \partial A))$.

Proof: Let $d = \operatorname{dist}(x_0, \partial A)$



WLOG $x_0 = 0$, $B(x_0, r) = r$.

Let $\varphi \in C^\infty(\mathbb{R}; [0,1])$ such that:
$\varphi(s) = 1$ if $s \le \frac{1}{2}$, $\varphi = 0$ if $s \ge 1$, $\varphi' \le 0$ on $\mathbb{R}$



Define:
$$\Phi(r) = \int_{\partial^* E} \varphi\left(\frac{|x|}{r}\right) d\mathcal{H}^{n-1}(x), \quad r \in (0,d)$$

$$T_r \in C_c^1(A; \mathbb{R}^n); \quad T_r(x) = \varphi\left(\frac{|x|}{r}\right) x, \quad x \in \mathbb{R}^n$$

Since E is stationary, then
$$\int_{\partial^* E} \operatorname{div}_E T_r = 0. \qquad (2)$$

We now proceed to compute $\operatorname{div}_E T_r$.

$$T_r(x) = \varphi\left(\frac{|x|}{r}\right)x, \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n$$
$$= \left(\varphi\left(\frac{|x|}{r}\right)x_1, \ldots, \varphi\left(\frac{|x|}{r}\right)x_n\right)$$

$$f_i(x) = \varphi\left(\frac{|x|}{r}\right)x_i, \quad |x| = (x_1^2 + \cdots + x_n^2)^{1/2}, \quad \frac{\partial}{\partial x_i}|x| = \frac{x_i}{|x|}$$

$$\nabla T_r(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_1} \\ & \vdots & \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

$$\operatorname{div} T_r = \operatorname{tr}(\nabla T_r) = \frac{\partial f_1}{\partial x_1} + \cdots + \frac{\partial f_n}{\partial x_n}$$

$$(*) \begin{cases} \dfrac{\partial f_i}{\partial x_i} = \varphi\left(\dfrac{|x|}{r}\right) + \dfrac{1}{r}\varphi'\left(\dfrac{|x|}{r}\right)\dfrac{x_i^2}{|x|} \\[2ex] \dfrac{\partial f_i}{\partial x_j} = \dfrac{1}{r}\varphi'\left(\dfrac{|x|}{r}\right)\dfrac{x_i x_j}{|x|}, \quad j \neq i \end{cases}$$

From $(*)$:

$$\boxed{\operatorname{div} T_r(x) = n\varphi\left(\frac{|x|}{r}\right) + \frac{1}{r}\varphi'\left(\frac{|x|}{r}\right)|x|} \quad (3)$$

Recall, from Lecture 22, Page 22.10, that:

$$\boxed{\operatorname{div}_E T_r(x) = \operatorname{div} T_r(x) - \nu_E(x) \cdot \left(\nabla T_r(x)\, \nu_E(x)\right)} \quad (4)$$

We now, need to compute $\nabla T_r(x)\nu_E(x)$. We think of the matrix $\nabla T_r(x)$ as a linear map acting on the vector $\nu_E(x)$.

Let $a_{ij}(x) = \frac{\partial f_i}{\partial x_j}(x)$

We write:

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}, \quad z = (z_1, \cdots, z_n), \quad c = (c_1, \cdots, c_n)$$

Note that:

$$c_i = \sum_{j=1}^{n} a_{ij}(x) z_j, \quad i = 1, \cdots, n$$

$$= \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(x) z_j$$

$$= \sum_{j=1}^{n} \left(\delta_{ij} \varphi\left(\frac{|x|}{r}\right) + \frac{1}{r} \varphi'\left(\frac{|x|}{r}\right) \frac{x_i x_j}{|x|} \right) z_j, \quad \delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$$

$$= \sum_{j=1}^{n} \delta_{ij} \varphi\left(\frac{|x|}{r}\right) z_j + \frac{|x|}{r} \varphi'\left(\frac{|x|}{r}\right) \left(\frac{x_j}{|x|} \cdot z_j \right) \frac{x_i}{|x|}$$

$$= \varphi\left(\frac{|x|}{r}\right) z_i + \frac{|x|}{r} \varphi'\left(\frac{|x|}{r}\right) \left(\frac{x}{|x|} \cdot z \right) \frac{x_i}{|x|}$$

We have shown:

$$\nabla T_r(x) z = c, \quad c_i = \varphi\left(\frac{|x|}{r}\right) z_i + \frac{|x|}{r} \varphi'\left(\frac{|x|}{r}\right) \left(\frac{x}{|x|} \cdot z \right) \frac{x_i}{|x|}$$

$$\Rightarrow c = \varphi\left(\frac{|x|}{r}\right) z + \frac{|x|}{r} \varphi'\left(\frac{|x|}{r}\right) \left(\frac{x}{|x|} \cdot z \right) \frac{x}{|x|}$$

Therefore:

$$\nabla T_r(x) z = \left(\varphi\left(\frac{|x|}{r}\right) \mathrm{Id} + \frac{|x|}{r} \varphi'\left(\frac{|x|}{r}\right) \frac{x}{|x|} \otimes \frac{x}{|x|} \right) z$$

Recall that the linear function $v \otimes w : \mathbb{R}^n \to \mathbb{R}^n$ is defined as $v \otimes w (z) = (w \cdot z) v, \ \forall z \in \mathbb{R}^n$.

We have shown then that:

$$\nabla T_r(x) = \varphi\left(\frac{|x|}{r}\right) Id + \frac{|x|}{r}\varphi'\left(\frac{|x|}{r}\right)\frac{x}{|x|}\otimes\frac{x}{|x|}$$

With this, we can now finish the computation of $\operatorname{div}_E T_r(x)$. Indeed,

$$\operatorname{div}_E T_r(x) = \operatorname{div} T_r(x) - \nu_E(x)\cdot\left(\nabla T_r(x)\nu_E(x)\right); \text{ by } (4)$$

$$= \operatorname{div} T_r(x) - \nu_E(x)\cdot\left(\varphi\left(\frac{|x|}{r}\right)\nu_E(x) + \frac{|x|}{r}\varphi'\left(\frac{|x|}{r}\right)\left(\frac{x}{|x|}\cdot\nu_E(x)\right)\frac{x}{|x|}\right)$$

$$= \operatorname{div} T_r(x) - \varphi\left(\frac{|x|}{r}\right) - \frac{|x|}{r}\varphi'\left(\frac{|x|}{r}\right)\left(\frac{x}{|x|}\cdot\nu_E(x)\right)\left(\frac{x}{|x|}\cdot\nu_E(x)\right)$$

$$= \operatorname{div} T_r(x) - \varphi\left(\frac{|x|}{r}\right) - \frac{|x|}{r}\varphi'\left(\frac{|x|}{r}\right)\frac{(x\cdot\nu_E(x))^2}{|x|^2}$$

$$= n\varphi\left(\frac{|x|}{r}\right) + \frac{1}{r}\varphi'\left(\frac{|x|}{r}\right)|x| - \varphi\left(\frac{|x|}{r}\right) - \frac{|x|}{r}\varphi'\left(\frac{|x|}{r}\right)\frac{(x\cdot\nu_E(x))^2}{|x|^2}; \text{ by } (3)$$

$$= (n-1)\varphi\left(\frac{|x|}{r}\right) + \frac{|x|}{r}\varphi'\left(\frac{|x|}{r}\right)\left(1 - \frac{(x\cdot\nu_E(x))^2}{|x|^2}\right)$$

Finally:

$$\boxed{\operatorname{div}_E T_r(x) = (n-1)\varphi\left(\frac{|x|}{r}\right) + \frac{|x|}{r}\varphi'\left(\frac{|x|}{r}\right)\left(1 - \frac{(x\cdot\nu_E(x))^2}{|x|^2}\right) \qquad (5)}$$

We now plug (5) in (3) to get: 23.7

$$\int_{\partial^* E}\left[(n-1)\varphi\left(\frac{|x|}{r}\right)+\frac{|x|}{r}\varphi'\left(\frac{|x|}{r}\right)-\frac{|x|}{r}\varphi'\left(\frac{|x|}{r}\right)\frac{(x\cdot\nu_E(x))^2}{|x|^2}\right]d\mathcal{H}^{n-1}=0$$

$\Longrightarrow$

$$(n-1)\int_{\partial^* E}\varphi\left(\frac{|x|}{r}\right)d\mathcal{H}^{n-1}+\int_{\partial^* E}\frac{|x|}{r}\varphi'\left(\frac{|x|}{r}\right)d\mathcal{H}^{n-1}$$

$$=\underbrace{\int_{\partial^* E}\frac{|x|}{r}\varphi'\left(\frac{|x|}{r}\right)\frac{(x\cdot\nu_E(x))^2}{|x|^2}d\mathcal{H}^{n-1}}_{\leq 0 \text{ since } \varphi'\leq 0}$$

$$\therefore\ (n-1)\Phi(r)+r\int_{\partial^* E}\frac{|x|}{r^2}\varphi'\left(\frac{|x|}{r}\right)d\mathcal{H}^{n-1}\leq 0;\quad \Phi(r)=\int_{\partial^* E}\varphi\left(\frac{|x|}{r}\right)$$

$$\therefore\ (n-1)\Phi(r)-r\Phi'(r)\leq 0 \qquad (6)$$

$$\Longrightarrow\ (r^{1-n}\Phi(r))'=r^{1-n}\Phi'(r)+\Phi(r)(1-n)r^{-n}$$

$$=\frac{\Phi'(r)}{r^{n-1}}+\frac{\Phi(r)(1-n)}{r^n}$$

$$=\frac{r\Phi'(r)+\Phi(r)(1-n)}{r^n}$$

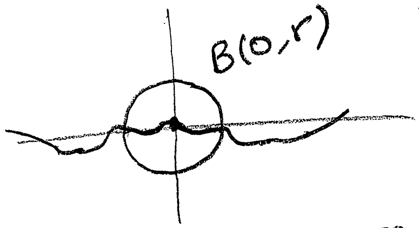
$$=-\frac{\Phi(r)(n-1)-r\Phi'(r)}{r^n}\geq 0;\ \text{by (6)}$$

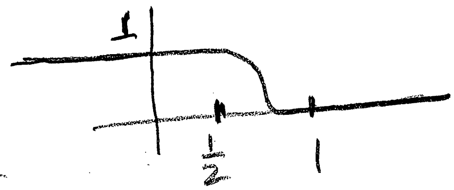
$\therefore\ (r^{1-n}\Phi(r))'\geq 0$ for $r\in(0,d)$ $\Rightarrow$ $r^{1-n}\Phi(r)$ is increasing on $r\in(0,d)$

$\forall\varphi\in C^\infty$ as defined at the beginning of proof.

We have

$r^{1-n}\int_{\partial^* E} \varphi\left(\frac{|x|}{r}\right) d\mathcal{H}^{n-1}$ increasing.





$\frac{|x|}{r} \le 1 \iff |x| \le r$

Let $\{\varphi_i\}_{i=1}^{\infty}$, $\varphi_i \in C^\infty(\mathbb{R};[0,1])$ s.t.

$$\varphi_i(z) \xrightarrow[i\to\infty]{} \chi_{(-\infty,1)}(z), \ \forall z \in \mathbb{R}.$$

monotonically

$\varphi_i \in C^\infty(\mathbb{R};[0,1])$

$\varphi_i \equiv 1, \ s \le \frac{1}{2}$

$\varphi_i \equiv 0, \ s \ge 1$

$\varphi_i' \le 0$.

By the Monotone-Convergence theorem:

$$\int_{\partial^* E} \varphi_i\left(\frac{|x|}{r}\right) d\mathcal{H}^{n-1} \xrightarrow[i\to\infty]{} \int_{\partial^* E} \chi_{(-\infty,1)}\left(\frac{|x|}{r}\right) d\mathcal{H}^{n-1}$$

$$= \int_{\partial^* E \cap B(0,r)} d\mathcal{H}^{n-1} \ ; \quad \text{since: } \frac{|x|}{r} \le 1 \iff |x| \le r$$

$$= P(E; B_r)$$

Let $\Phi_i(r) = \int_{\partial^* E} \varphi_i\left(\frac{|x|}{r}\right) d\mathcal{H}^{n-1}$

$\Longrightarrow$ $r^{1-n}\Phi_i(r)$ is increasing or $r \in (0,d)$, $\forall i$.

$\Longrightarrow$ $r^{1-n}\Phi_i(r) \xrightarrow{i\to\infty} r^{1-n}P(E;B_r)$ monotonically.

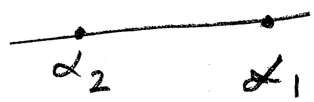
Therefore, $r^{1-n}P(E;B_r)$ is increasing on $r \in (0,d)$.

Indeed, let $r_1 < r_2$, $r_1, r_2 \in (0,d)$

$$\alpha_1 = r_1^{1-n} P(E; B_{r_1}), \quad \alpha_2 = r_2^{1-n} P(E; B_{r_2})$$

We have $\alpha_1 \le \alpha_2$; since otherwise, if

$$\alpha_1 > \alpha_2$$



Since $r_1^{1-n} \Phi_i(r_1) \to \alpha_1$ monotonically increasing then $\alpha_2 < r_1^{1-n} \Phi_i(r_1) < \alpha_1$ for i large enough. Since $r_1^{1-n} \Phi_i(r_1) \le r_2^{1-n} \Phi_i(r_2)$, we have:

$$\alpha_2 < r_2^{1-n} \Phi_i(r_2); \quad \text{large } i,$$

which is not possible since $r_2^{1-n} \Phi_i(r_2) \longrightarrow \alpha_2$ monotonically increasing.

Thus, we have proved:

$$r^{1-n} P(E; B_r) \text{ is increasing on } r \in (0,d) \quad \blacksquare$$

We have the following Corollary of Theorem 1:

Corollary 1 (<u>Density estimates for stationary sets</u>): If E is stationary for perimeter in the open set A, then:

$$P(E; B(x,r)) \geq \omega_{n-1} r^{n-1}, \quad \forall\, x \in A \cap \partial E, \; B(x,r) \subset A$$

In particular:

$$\mathcal{H}^{n-1}(A \cap (\partial E \setminus \partial^* E)) = 0.$$

Moreover, $\Theta_{n-1}(\mu)(x)$ (with $\mu = \mathcal{H}^{n-1} \llcorner \partial^* E$) exists on $A \cap \partial E$, and

(a) $\Theta_{n-1}(\mu)(x) := \lim\limits_{r \to 0} \dfrac{\mathcal{H}^{n-1}(\partial^* E \cap B(x,r))}{\omega_{n-1} r^{n-1}} \geq 1 \quad \forall\, x \in A \cap \partial E$

(b) $\Theta_{n-1}(\mu)(x) = 1, \quad \forall\, x \in A \cap \partial^* E.$

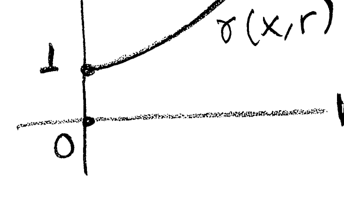
<u>Proof</u>: By Theorem 1, we consider the increasing function:

$$\gamma(x,r) = \frac{P(E; B(x,r))}{\omega_{n-1} r^{n-1}}, \quad x \in A, \quad r < \operatorname{dist}(x, \partial A)$$

We have proved in another Lecture that:

$$\Theta_{n-1}(\mu)(x) = \gamma(x, 0^+) = 1, \quad \text{if } x \in A \cap \partial^* E$$

$$\Rightarrow \boxed{\gamma(x,r) \geq 1, \quad \forall\, x \in A \cap \partial^* E, \; B(x,r) \subset A}$$



Let $x \in A \cap (\partial E \setminus \partial^* E)$

23.11

Since $\overline{\partial^* E} = \partial E$ (recall that we are working with the representative E such that $\operatorname{spt} \mu_E = \partial E$). Then, $\exists x_i$, $x_i \in A \cap \partial^* E$ such that:

$$x_i \to x.$$

(**) We need the following result (see Exercise 4.27 in textbook):

Let $u(x) = \mu(\overline{B}(x,r))$, $x \in \mathbb{R}^n$, μ Radon measure

$\Rightarrow u$ is upper semicontinuos

With $\mu = \mathcal{H}^{n-1} \llcorner (A \cap \partial^* E)$, we apply (**) to get:

$$P(E; \overline{B(x,r)}) \geq \limsup_{i \to \infty} P(E; \overline{B(x_i, r)})$$

$$\geq \omega_{n-1} r^{n-1} \quad ; \quad \text{since } \gamma(x_i, r) \geq 1;$$

because $x_i \in \partial^* E$

Now:

$$P(E; B(x,s)) \geq \limsup_{\varepsilon \to 0} P(E; \overline{B(x, s-\varepsilon)}) \quad ; \quad \text{Since } \overline{B(x, s-\varepsilon)} \subset \overline{B(x,s)}$$

$$\geq \limsup_{\varepsilon \to 0} \omega_{n-1} (s-\varepsilon)^{n-1} \quad ; \quad \text{by the above computation.}$$

$$= \omega_{n-1} s^{n-1}$$

We have shown: $P(E; B(x,r)) \geq \omega_{n-1} r^{n-1}$, $\forall x \in \partial E \cap A$, $B(x,r) \subset A$.

Again, set:

$$\mu = \mathcal{H}^{n-1} \llcorner (A \cap \partial^* E).$$

Hence: $$\boxed{\Theta^*_{n-1}(\mu) \geq 1 \quad \forall x \in A \cap \partial E} \quad (7)$$

From a Theorem proved in Lecture 7 (Page 7.2),

it follows from (7) that:

$$\mu(A \cap \partial E) \geq \mathcal{H}^{n-1}(A \cap \partial E)$$

$$\therefore \quad \mathcal{H}^{n-1}(A \cap \partial^* E) \geq \mathcal{H}^{n-1}(A \cap \partial E)$$

And, since clearly, $\mathcal{H}^{n-1}(A \cap \partial^* E) \leq \mathcal{H}^{n-1}(A \cap \partial E)$,

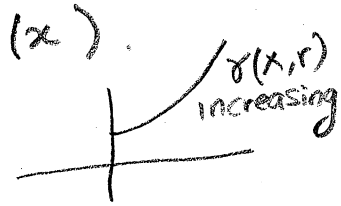
we conclude that:

$$\mathcal{H}^{n-1}(A \cap \partial^* E) = \mathcal{H}^{n-1}(A \cap \partial E);$$

that is: $$\boxed{\mathcal{H}^{n-1}(A \cap (\partial E \setminus \partial^* E)) = 0.}$$

Now, notice that $\Theta_{n-1}(\mu)(x)$ exists at every $x \in A \cap \partial E$, since our Theorem 1 implies:

$$\inf_{r>0} \gamma(x,r) = \lim_{r \to 0^+} \gamma(x,r) = \Theta_{n-1}(\mu)(x).$$



Moreover; $\Theta_{n-1}(\mu)(x) \geq 1$, $\forall x \in A \cap \partial E$,
since $P(E; B(x,r)) \geq \omega_{n-1} r^{n-1}$ $\forall x \in A \cap \partial E$.

In particular, if $x \in A \cap \partial^* E$, we showed in a previous lecture that $\Theta_{n-1}(\mu)(x) = 1$ ∎