Lecture 24

Tangential differentiability, Area formula on rectifiable sets, Coarea formula on $\mathcal{H}^{n-1}$-rectifiable sets.

<u>Def</u>: Let M be a k-dimensional C^1-surface in $\mathbb{R}^n$ and let $x \in M$. A function $f: \mathbb{R}^n \to \mathbb{R}^m$ is tangentially differentiable with respect to M at x if there exists a linear function:

$$\nabla^M f(x) \in \mathbb{R}^m \otimes T_x M$$

such that, uniformly on $\{v \in T_x M : |v| = 1\}$,

$$\lim_{h \to 0} \frac{f(x+hv) - f(x)}{h} = \nabla^M f(x)\, v$$

In other words:
The restriction of f to $x + T_x M$ is differentiable at x.

<u>Def</u>: The tangential Jacobian of f with respect to M at x is then defined by:

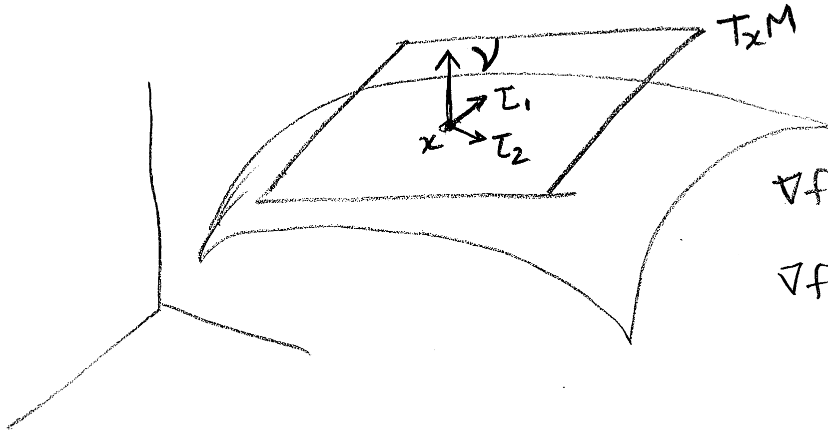
$$J^M f(x) = \sqrt{\det\left(\nabla^M f(x)^* \, \nabla^M f(x)\right)}$$

<u>Remark</u>: $f \in C^1(\mathbb{R}^n; \mathbb{R}^m)$, M C^1-surface, $x \in M \Rightarrow f$ is tangentially differentiable at x, and $\nabla^M f(x) = \nabla f(x) \llcorner T_x M$. Thus, if $\{\tau_i\}_{i=1}^k$ and $\{\nu_j\}_{j=1}^{n-k}$ are orthonormal basis of $T_x M$ and $(T_x M)^\perp$ resp., then:

$$\nabla^M f(x) = \sum_{i=1}^k (\nabla f(x) \tau_i) \otimes \tau_i = \nabla f(x) - \sum_{j=1}^{n-k} (\nabla f(x) \nu_j) \otimes \nu_j$$

Ex: $f: \mathbb{R}^3 \to \mathbb{R}$

M an $(n-1)$-dimensional surface.



$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}(x), \frac{\partial f}{\partial x_2}(x), \frac{\partial f}{\partial x_3}(x)\right)$

$\nabla f(x): \mathbb{R}^3 \to \mathbb{R}$ linear map.

f is tangentially differentiable at x if $\exists \nabla^M f: T_xM \to \mathbb{R}$ linear such that:

$$(*) \quad \lim_{h\to 0} \frac{f(x+hv) - f(x) - \nabla^M f(x)(hv)}{h} = 0, \quad \forall v \in T_xM, |v|=1$$

which is the definition of differentiability of a function of two variables (see Calculus book).

Thus, f tangentially diff. at x means:

$$f \llcorner T_xM \text{ is differentiable}$$

Now, the linear map $\nabla f(x): \mathbb{R}^3 \to \mathbb{R}$ can be written as:

$$\nabla f(x) = \sum_{i=1}^{2} (\nabla f(x)\tau_i) \otimes \tau_i + (\nabla f(x)\nu) \otimes \nu,$$

where recall that if $v, w \in \mathbb{R}^n$, the linear map $w \otimes v$ is defined as $(w \otimes v)(z) = (v \cdot z)w \quad \forall z \in \mathbb{R}^n$. Indeed:

$$\nabla f(x) z = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i}(x) z_i, \quad z = z_1(1,0) + z_2(0,1).$$

Write z is the basis $\{\tau_1, \tau_2, \nu\}$ as:

$$z = \tilde{z}_1 \tau_1 + \tilde{z}_2 \tau_2 + \tilde{z}_3 \nu.$$

Then:

$$\left(\sum_{i=1}^{2} (\nabla f(x)\tau_i) \otimes \tau_i\right) z + \left((\nabla f(x)\nu) \otimes \nu\right) z$$

$$= \sum_{i=1}^{2} (z \cdot \tau_i)\ \nabla f(x)\tau_i + (\nu \cdot z)\ \nabla f(x)\nu$$

$$= \sum_{i=1}^{2} \tilde{z}_i \nabla f(x)\tau_i + \tilde{z}_3 \nabla f(x)\nu$$

$$= \sum_{i=1}^{2} \nabla f(x)(\tilde{z}_i \tau_i) + \nabla f(x)\tilde{z}_3 \nu$$

$$= \nabla f(x)\left(\tilde{z}_1\tau_1 + \tilde{z}_2\tau_2 + \tilde{z}_3\nu\right)$$

$$= \nabla f(x) z, \quad \forall z \in \mathbb{R}^3$$

$$\therefore \boxed{\nabla f(x) = \sum_{i=1}^{2} (\nabla f(x)\tau_i) \otimes \tau_i + \nabla f(x)\nu \otimes \nu \text{ as linear maps.}} \quad (1)$$

If $f: \mathbb{R}^3 \to \mathbb{R}$ is $C^1 \Rightarrow f$ is differentiable at x.

That is:

$$\boxed{\lim_{y \to x} \frac{f(y) - f(x) - \nabla f(x)(y - x)}{\|y - x\|} = 0} \quad (**)$$

The existence of the limit in $(**)$ implies the existence of the limit in $(*)$ and:

$$\nabla^M f(x) = \nabla f(x) \llcorner T_x M$$

; by uniqueness of the linear map in $(*)$.

If $z \in T_x M$, $z \in \mathbb{R}^3$, then $\exists\, \tilde{z}_1, \tilde{z}_2$ such that $z = \tilde{z}_1\tau_1 + \tilde{z}_2\tau_2 + 0\nu$. Therefore, using (1) above:

$$\nabla^M f(x)(z) = \nabla f(x) \llcorner T_x M(z) = \nabla f(x) z = \left(\sum_{i=1}^{2} (\nabla f(x)\tau_i) \otimes \tau_i\right)(z)$$

$$\Rightarrow \nabla^M f(x) = \nabla f(x) \llcorner T_x M(z) = \sum_{i=1}^{2} (\nabla f(x)\tau_i) \otimes \tau_i = \nabla f(x) - (\nabla f(x)\nu) \otimes \nu, \text{ as linear maps.}$$

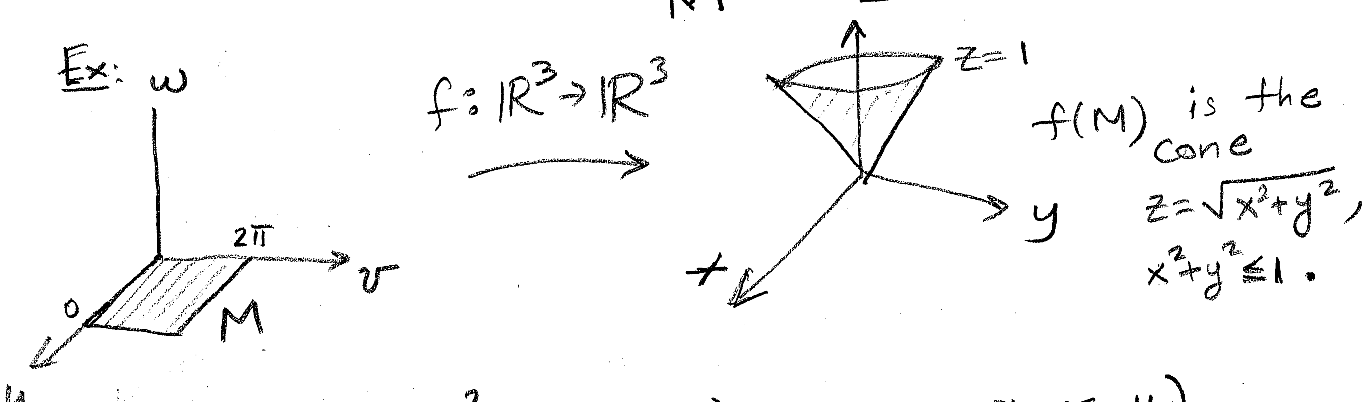
Theorem: (Area formula for rectifiable sets).

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ Lipschitz, $1 \le k \le m$.

If f is injective on M, then:

$$\mathcal{H}^k(f(M)) = \int_M J^M f \, d\mathcal{H}^k$$

Ex:



$M = \{(u,v,0) : 0 \le u \le 1, 0 < v \le 2\pi\}$, $f(u,v,w) = (u\cos v, u\sin v, u)$

$$\mathcal{H}^2(f(M)) = \int_M J^M f(u,v,w) \, d\mathcal{H}^2$$

$$\nabla f(u,v,w) = \begin{pmatrix} \cos v & -u\sin v & 0 \\ \sin v & u\cos v & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\nabla^M f(u,v,w) = \nabla f(u,v,w) \, \llcorner \, T_{(u,v,w)} M = \nabla f(u,v,w) \, \llcorner \, \mathbb{R}^2.$$

Hence:
$$\nabla^M f(u,v,w) : \mathbb{R}^2 \to \mathbb{R}^3$$

In matrix form, $\nabla^M f$ is the 3×2 matrix:

$$\nabla^M f(u,v,w) = \begin{pmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \\ 1 & 0 \end{pmatrix}$$

Hence

$$\nabla^M f(u,v,w)^* = \begin{pmatrix} \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0 \end{pmatrix}$$

Hence:

$$\nabla^M f(u,v,w)^* \nabla^M f(u,v,w) = \begin{pmatrix} \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0 \end{pmatrix} \begin{pmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 \\ 0 & u^2 \end{pmatrix}$$

$$\therefore J^M f(u,v,w) = \sqrt{\det\left(\nabla^M f(u,v,w)^* \nabla^M f(u,v,w)\right)}$$

$$= \sqrt{\det\begin{pmatrix} 2 & 0 \\ 0 & u^2 \end{pmatrix}}$$

$$= \sqrt{2u^2} = \sqrt{2}\,u .$$

$$\therefore \mathcal{H}^2(f(M)) = \mathcal{H}^2(\text{Cone}) = \int_M \sqrt{2}\,u \; d\mathcal{H}^2(u,v,w)$$

$$= \int_0^1 \int_0^{2\pi} \sqrt{2}\,u \; dv\,du, \quad M = [0,1]\times[0,2\pi]$$

$\mathcal{H}^2 = \mathcal{L}^2$ on $\mathbb{R}^2$,

$$= \sqrt{2}\,2\pi \left[\frac{u^2}{2}\right]_0^1$$

$$= \sqrt{2}\,(\cancel{2}\pi)\cdot\frac{1}{\cancel{2}} = \boxed{\sqrt{2}\,\pi}$$

Note: If M is a locally $\mathcal{H}^k$-rectifiable set, and $f: \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz map, then $\nabla^M f(x)$ exists at $\mathcal{H}^k$-a.e. $x \in M$ (see Textbook, Theorem 11.4).

In particular for $u: \mathbb{R}^n \to \mathbb{R}$ Lipschitz, from previous note, u is tangentially differentiable at a $\mathcal{H}^{n-1}$-a.e. point of a locally $\mathcal{H}^{n-1}$ rectifiable set M. From Remark on Page 24.1 we have:

$$\nabla^M u(x) = \nabla u(x) - (\nabla u(x) \cdot \nu(x))\, \nu(x),$$

whenever ∇u and $\nabla^M u$ exists at x, and $\nu(x) \in S^{n-1}$ is such that $\nu(x)^\perp = T_x M$. Here, we think of the previous identity as equality of vectors (instead of linear maps).

We have:

Theorem (Coarea formula on locally $(n-1)$-rectifiable sets).
If M is a locally $\mathcal{H}^{n-1}$ rectifiable set in $\mathbb{R}^n$ and $u: \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz function, then:

$$\int_{\mathbb{R}^n} \mathcal{H}^{n-2}(M \cap \{u = t\})\, dt = \int_M |\nabla^M u| \, d\mathcal{H}^{n-1}$$

In particular, if $g: M \to [-\infty, \infty]$ is a Borel function and either $g \geq 0$ or $g \in L^1(\mathbb{R}^n, \mathcal{H}^{n-1} \llcorner M)$, then

$$\int_{\mathbb{R}} \left(\int_{M \cap \{u=t\}} g \, d\mathcal{H}^{n-2} \right) dt = \int_M g \, |\nabla^M u| \, d\mathcal{H}^{n-1}.$$

Using the coarea formula for rectifiable sets, we now give a different proof of the monotonicity of density ratios (Theorem 1, Lecture 23, Page 23.3). This new proof is more intuitive, and it requires E to be a minimizer, while the previous proof requires E to be stationary.

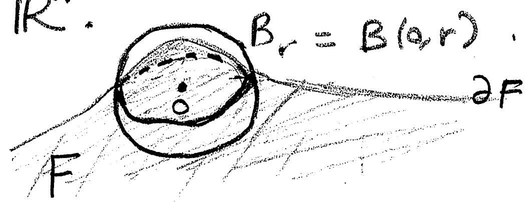
<u>Theorem 2</u>: If E is a minimizer in the open set A and $x_0 \in A \cap \partial E$, then the function

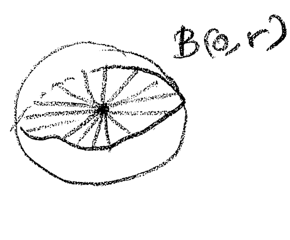
$$\frac{P(E; B(x_0,r))}{r^{n-1}}$$

is increasing on $r < \min(0, \mathrm{dist}(x_0, \partial A))$

<u>Proof</u>: WLOG $x_0 = 0$

<u>Step one</u>: Let F be an open set with smooth boundary in $\mathbb{R}^n$.

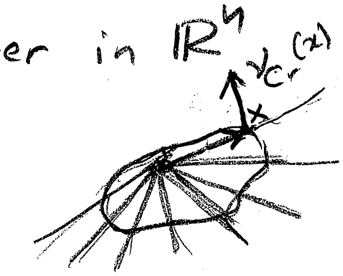




$\forall r > 0$, $F \cap \partial B_r$ is relatively open in ∂B_r. By Sard's lemma, $\partial F \cap \partial B_r$ is a smooth $(n-2)$-dimensional surface for a.e. $r > 0$. We now consider the cone C_r with vertex at 0 and passing through $F \cap \partial B_r$; that is,

$$C_r = \{\lambda y : \lambda > 0,\ y \in F \cap \partial B_r\}$$

We have (see Lemma 24.6 in textbook) that for a.e. $r>0$, the cone C_r is of locally finite perimeter in $\mathbb{R}^n$ with



$$\mu_{C_r} = \nu_{C_r} \mathcal{H}^{n-1} \llcorner \partial C_r$$

$$\nu_{C_r}(x) \cdot x = 0 \quad \forall x \in C_r \setminus \{0\}$$

We apply the coarea formula to the $\mathcal{H}^{n-1}$-rectifiable set $M = \partial C_r$, with the Lipschitz function $u(x) = |x|$, $x \in \mathbb{R}^n$, and we notice that:

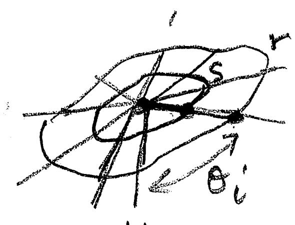
$$\nabla^M u(x) = \nabla u(x) - (\nabla u(x) \cdot \nu_{C_r}(x))\, \nu_{C_r}(x)$$

$$= \frac{x}{|x|} - \underbrace{\left(\frac{x}{|x|} \cdot \nu_{C_r}(x)\right)}_{0} \nu_{C_r}(x)$$

$\therefore \quad |\nabla^M u(x)| = 1$

$\therefore \quad \displaystyle\int_0^r \mathcal{H}^{n-2}(\partial C_r \cap \partial B_s)\, ds = \int_{B_r \cap \partial C_r} |\nabla^M u|\, d\mathcal{H}^{n-1} = P(C_r; B_r)$

Notice that:



$$s = \left(\frac{s}{r}\right) r$$

We want to compare $\mathcal{H}^{n-2}(\partial C_r \cap \partial B_s)$ with $\mathcal{H}^{n-2}(\partial F \cap \partial B_r)$. If $n=3$, $n-2=1$ and notice that

$$\mathcal{H}^1(\partial C_r \cap \partial B_s) = \lim_{N\to\infty} \sum_{i=1}^N s\theta_i$$

$$= \lim_{N\to\infty} \sum_{i=1}^N \left(\frac{s}{r}\right) r\theta_i = \frac{s}{r} \lim_{N\to\infty} \sum_{i=1}^N r\theta_i$$

$$= \frac{s}{r} \mathcal{H}^1(\partial F \cap \partial B_r).$$

Thus:

$$P(C_r;B_r) = \int_0^r \mathcal{H}^{n-2}(\partial C_r \cap \partial B_s)\,ds$$

$$= \int_0^r \left(\frac{s}{r}\right)^{n-2} \mathcal{H}^{n-2}(\partial F \cap \partial B_r)\,ds$$

$$= \mathcal{H}^{n-2}(\partial F \cap \partial B_r) \cdot \frac{1}{r^{n-2}} \left[\frac{s^{n-1}}{n-1}\right]_0^r$$

$$= \frac{\mathcal{H}^{n-2}(\partial F \cap \partial B_r)}{r^{n-2}} \cdot \frac{r^{n-1}}{n-1}$$

$$= \frac{\mathcal{H}^{n-2}(\partial F \cap \partial B_r) \cdot r}{n-1}$$

We have proved:

$$\boxed{P(C_r;B_r) = \frac{r\,\mathcal{H}^{n-2}(\partial F \cap \partial B_r)}{n-1}} \quad (2).$$

Step two: Let $\{F_i\}_{i=1}^{\infty}$ open sets with smooth boundary such that:

$$F_i \to E \text{ in } L^1_{loc}(\mathbb{R}^n), \qquad |\mu_{F_i}| \xrightarrow{*} |\mu_E|$$

Choose r such that:

Note: This properties hold for a.e. r.

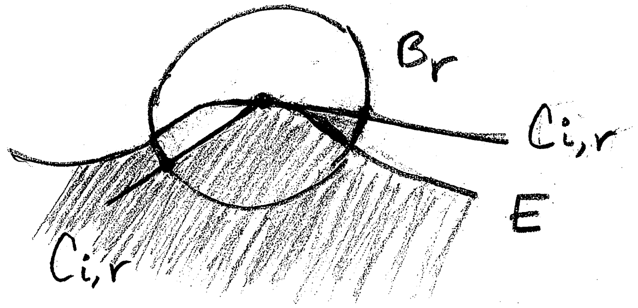
$$\begin{cases} (a)\ \mathcal{H}^{n-1}(\partial^* E \cap \partial B_r) = 0 \\ (b)\ \partial B_r \cap \partial F_i \text{ is an } (n-2)\text{-dimensional smooth surface.} \end{cases}$$

Define $C_{i,r}$ as in step one:

$$C_{i,r} = \{\lambda y : \lambda > 0,\ y \in F_i \cap \partial B_r\}.$$

Define:

$$E_i = (C_{i,r} \cap B_r) \cup (E \setminus B_r)$$



Since E is a minimizer we have (see Lecture 21, Page 21.10):

$$P(E;B_r) \le P(E_i;B_r) \le P(C_{i,r};B_r) + \mathcal{H}^{n-1}(\partial B_r \cap (E^{(1)} \Delta C_{i,r}))$$

Thus, using (2):

$$\boxed{P(E;B_r) \le \frac{r\mathcal{H}^{n-2}(\partial F_i \cap \partial B_r)}{n-1} + \mathcal{H}^{n-1}(\partial B_r \cap (E^{(1)} \Delta F_i)) \quad \text{for a.e. } r \in (0,d),\ d = \mathrm{dist}(x_0, \partial A)} \qquad (3)$$

We now integrate on an interval $(s,t) \subset (0,d)$:

$$\int_s^t P(E;B_r) \le \frac{1}{n-1}\int_s^t r\mathcal{H}^{n-2}(\partial F_i \cap \partial B_r)\,dr + \int_s^t \mathcal{H}^{n-1}(\partial B_r \cap (E^{(1)} \Delta F_i))\,dr$$

$$\le \frac{1}{n-1}\int_s^t r\mathcal{H}^{n-2}(\partial F_i \cap \partial B_r)\,dr + |(E^{(1)} \Delta F_i) \cap B_d|$$

By the coarea formula for rectifiable sets (see page 24.6, with $g = |x|$ and $M = \partial F_i$):

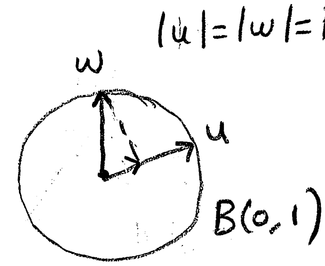
$$\boxed{\int_s^t \left(\int_{\partial F_i \cap \{u=r\}} |x|\, d\mathcal{H}^{n-2}\, dr \right) = \int_{\partial F_i \cap (B_t \setminus \overline{B}_s)} |x|\, |\nabla^{\partial F_i} u|\, d\mathcal{H}^{n-1}} \qquad (4)$$

We have $|\nabla^{\partial F_i} u| = |\nabla u| \leq 1$. Indeed,

$$\nabla^{\partial F_i} u(x) = \nabla u(x) - (\nu_{F_i}(x) \cdot \nabla u)\, \nu_{F_i}$$

$$|\nabla u(x)| = \left|\frac{x}{|x|}\right| = 1 \Rightarrow$$

$$|\nabla^{\partial F_i} u(x)|^2 = 1 - \underbrace{\left(\nu_{F_i}(x) \cdot \frac{x}{|x|}\right)^2}_{\leq 1} \leq 1$$



Since this is the length of the projection of a unit vector onto another unit vector.

Therefore, from (4):

$$\int_s^t r\,\mathcal{H}^{n-2}(\partial F_i \cap \partial B_r)\,dr = \int_{\partial F_i \cap (B_t \setminus \overline{B}_s)} |x|\,|\nabla^{\partial F_i} u|\,d\mathcal{H}^{n-1}$$

$$\leq t\,\mathcal{H}^{n-1}(\partial F_i \cap (B_t \setminus \overline{B}_s))$$

$$= t\,P(F_i;\, B_t \setminus \overline{B}_s)$$

Thus,

$$\int_s^t P(E;B_r)\,dr \leq \frac{t\,P(F_i;\, B_t \setminus \overline{B}_s)}{n-1} + |(E^{(1)} \Delta F_i) \cap B_d|$$

Letting $i \to \infty$ yields:

$$\int_s^t P(E;B_r)\,dr \leq t\,P(E;\, B_t \setminus \overline{B}_s)$$

Finally, if s is a point of differentiability for $r \mapsto P(E;B_r)$ we have:

$$\frac{1}{t-s}\int_s^t P(E;B_r)\,dr \leq t\,\frac{P(E;\, B_t \setminus \overline{B}_s)}{t-s} = \frac{t}{n-1}\left[\frac{P(E;B_t) - P(E;B_s)}{t-s}\right]$$

Letting $t \to s^+$ yields:

$$P(E; B_s) \le \frac{s}{n-1}\left(\frac{d}{dr}(P(E; B_r))\right)(s)$$

Let $\Phi(r) = P(E; B_r)$

We have shown that:

$$\Phi(r) \le \frac{r}{n-1}\Phi'(r) \quad \text{for a.e. } 0<r<d$$

$$\iff r^{-n}\Phi(r) \le \frac{r^{1-n}}{n-1}\Phi'(r)$$

$$\iff (n-1)r^{-n}\Phi(r) \le r^{1-n}\Phi'(r)$$

$$\iff (1-n)r^{-n}\Phi(r) + r^{1-n}\Phi'(r) \ge 0$$

$$\iff \left(r^{1-n}\Phi(r)\right)' \ge 0 \quad \text{for a.e. } r.$$

$$\iff \frac{d}{dr}\left[r^{1-n}P(E; B_r)\right] \ge 0 \quad \text{for a.e. } r.$$

The non-rigorous understanding of the monotonicity:

E minimizer

$B(0,r)$

Cone C_r

E

$\partial^* E$ not smooth

$$P(E;B_r) \le P(C_r) \approx \frac{r\,\mathcal{H}^{n-2}(\partial^* E \cap \partial B_r)}{n-1}$$

$$\therefore \quad \boxed{P(E;B_r) \le \frac{r}{n-1}\,\mathcal{H}^{n-2}(\partial^* E \cap \partial B_r) \quad (1)}$$

Let $\Phi(r) = P(E;B_r)$. Define also:

$$\alpha(r) = \int_0^r \mathcal{H}^{n-2}(\partial^* E \cap \partial B_s)\,ds$$

$$\therefore \quad \alpha'(r) = \mathcal{H}^{n-2}(\partial^* E \cap \partial B_r); \quad \alpha(r) \text{ is increasing}$$

On the other hand:

$$\Phi(r) = P(E;B_r) \approx \int_{\partial^* E \cap B_r} |\nabla^{\partial^* E} u|\,d\mathcal{H}^{n-1}; \quad \text{since } |\nabla^{\partial^* E} u| \le 1, \text{ because } |\nabla u| = \left|\frac{x}{|x|}\right| = 1$$

$$\| \text{ coarea}$$

$$\int_0^r \mathcal{H}^{n-2}(\partial^* E \cap \partial B_s)\,ds$$

$$\therefore \quad \Phi(r) \approx \alpha(r)$$

$$\therefore \quad \Phi'(r) \approx \alpha'(r) = \mathcal{H}^{n-2}(\partial^* E \cap \partial B_r)$$

From (1): $\quad P(E;B_r) \le \frac{r}{n-1}\,\Phi'(r) \implies \Phi(r) \le \frac{r}{n-1}\,\Phi'(r)$, for a.e. r.