Lecture 25

Lipschitz Graph Criterion.

We will study the regularity of the minimizer in two steps:

1.- Regularity of $\partial^* E \cap A$

2.- Estimate of the size of the singular set $\Sigma = A \cap (\partial E \setminus \partial^* E)$

<u>Step 1</u>: Regularity of $\partial^* E$ in A

E minimizer in A

$P(E) \leq P(F)$ $\forall F$, $E \Delta F \subset\subset A$.

We have shown the following properties of E:

(a) E is stationary;

$$\int_{\partial^* E} \operatorname{div}_E T \, d\mathcal{H}^{n-1} = 0 \qquad \forall T \in C_c^\infty(A, \mathbb{R}^n)$$

(b) Uniform density estimates for E:

$$\frac{1}{2^n} \leq \frac{|E \cap B(x,r)|}{\omega_n r^n} \leq 1 - \frac{1}{2^n}, \quad c(n) \leq \frac{P(E; B(x,r))}{r^{n-1}} \leq n\omega_n,$$

for all $x \in \partial E \cap A$, $\forall r \leq \operatorname{dist}(x, \partial A)$. These estimates imply the weak regularity $\mathcal{H}^{n-1}(A \cap (\partial E \setminus \partial^* E)) = 0$.

(c) Monotonicity of density ratios:

$$r \longmapsto \frac{P(E, B(x,r))}{r^{n-1}}$$

is increasing $\forall x \in \partial E \cap A$.

(d) Existence of densities $\theta_{n-1}(\mu)(x)$, for every $x \in \partial E \cap A$, with $\mu = \mathcal{H}^{n-1} \llcorner \partial^* E$.

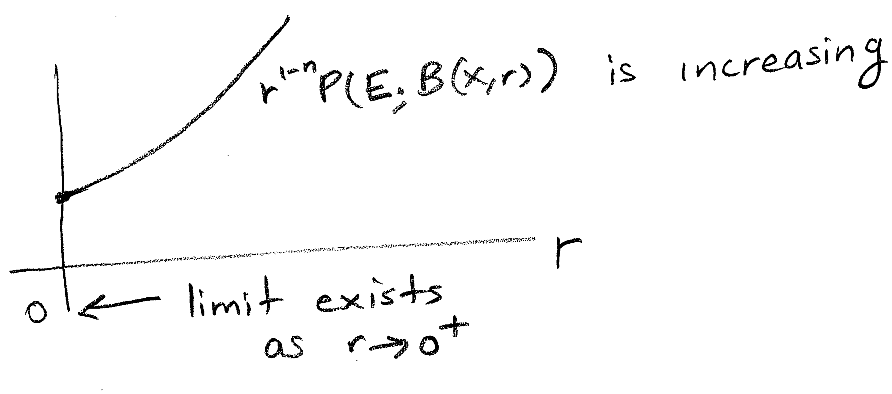
$$\lim_{r \to 0^+} \frac{\mu(B(x,r))}{\omega_{n-1} r^{n-1}} = \lim_{r \to 0^+} \frac{\mathcal{H}^{n-1}(\partial^* E \cap B(x,r))}{\omega_{n-1} r^{n-1}}$$

$$= \lim_{r \to 0^+} \frac{P(E; B(x,r))}{\omega_{n-1} r^{n-1}}$$ exists, for every $x \in \partial E \cap A$.

Moreover:

$\theta_{n-1}(\mu)(x) \geq 1 \quad \forall x \in A \cap \partial E$

$\theta_{n-1}(\mu)(x) = 1 \quad \forall x \in A \cap \partial^* E$



We will prove the following:

Theorem: $A \cap \partial^* E$ is an $(n-1)$-dimensional analytic manifold relatively open in $A \cap \partial E$.

It will take several classes to prove this theorem.

We start with the following:

Theorem 1 (Lipschitz graph criterion). If E is a set of locally finite perimeter in $\mathbb{R}^n$ with $\operatorname{spt}\mu_E = \partial E$ and $0 \in \partial E$, and if $u : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function with $\operatorname{Lip}(u) \le 1$, such that

$$C(0,1) \cap \partial^* E \subset \{(z, u(z)) : z \in \mathbb{R}^{n-1}\} \qquad (1)$$

then

$$C(0,1) \cap \partial E = \{(z, u(z)) : z \in D(0,1)\} \qquad (2)$$

and either

$$C(0,1) \cap E = \{(z,t) \in C(0,1) : z \in D(0,1),\ -1 < t < u(z)\} \qquad (3)$$

or

$$C(0,1) \cap E = \{(z,t) : z \in D(0,1),\ u(z) < t < 1\} \qquad (4)$$

Moreover, for every Borel set $G \subset D(0,1)$, we have:

$$P(E, C(0,1) \cap p^{-1}(G)) = \int_G \sqrt{1 + |\nabla' u(z)|^2}\, dz, \qquad (5)$$

and, depending on which (1) or (2) holds true, we have either

$$\nu_E(z, u(z)) = \frac{(\nabla' u(z), -1)}{\sqrt{1 + |\nabla' u(z)|^2}} \qquad (6)$$

or

$$\nu_E(z, u(z)) = \frac{(-\nabla' u(z), 1)}{\sqrt{1 + |\nabla' u(z)|^2}}$$

for a.e. $z \in D(0,1)$

We now clarify the notation in previous theorem:

NOTATION:

$$\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$$

$$p : \mathbb{R}^n \to \mathbb{R}^k \times \{0\}, \quad q : \mathbb{R}^n \to 0 \times \mathbb{R}^{n-k} = \mathbb{R}^{n-k}.$$

are the horizontal and vertical projections, so that

$$x = (px, qx), \quad x \in \mathbb{R}^n.$$

The cylinder of center $x \in \mathbb{R}^n$ and radius $r > 0$,

$$C(x,r) = \{y \in \mathbb{R}^n : |p(y-x)| < r, \ |q(x-r)| < r\},$$

and the k-dimensional ball of center $z \in \mathbb{R}^k$ and radius $r > 0$ is:

$$D(z,r) = \{w \in \mathbb{R}^k : |z-w| < r\}$$

Moreover, we sometimes abbreviate:

$$C(0,r) = C_r, \quad C_1 = C, \quad D(0,r) = D_r \quad D_1 = D$$

When $k = n-1$, we write $px = x'$, $qx = x_n$, so that $x = (x', x_n)$. The gradient operator in $\mathbb{R}^n$ and in $\mathbb{R}^{n-1}$ is denoted by ∇ and $\nabla' = (\partial_1, \dots, \partial_{n-1})$. Thus, if $u : \mathbb{R}^n \to \mathbb{R}$ has gradient $\nabla u(x) \in \mathbb{R}^n$ at $x \in \mathbb{R}^n$, then we set $\nabla u(x) = (\nabla' u(x), \partial_n u(x))$. Moreover,

$$C(x,r,\nu) = x + \{y \in \mathbb{R}^n : |y \cdot \nu| < r, \ |y - (y \cdot \nu)\nu| < r\},$$

where $x \in \mathbb{R}^n$, $r > 0$, and $\nu \in S^{n-1}$!

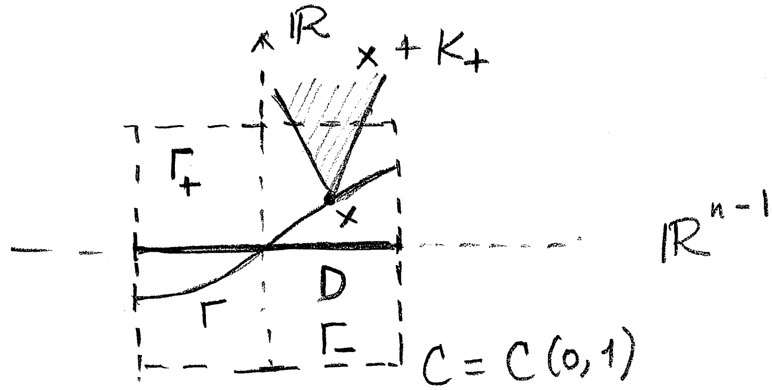
Proof of Theorem 1:

Define:

$\Gamma_+ = \{(z,t) : z \in D,\ u(z) < t < 1\}$

$\Gamma_- = \{(z,t) : z \in D,\ -1 < t < u(z)\}$

$\Gamma = \{(z, u(z)) : z \in D\}$



Since Γ is closed and $\partial^* E \subset \Gamma$ then:

$$\partial E = \overline{\partial^* E} = \operatorname{spt} \mu_E \subset \Gamma$$

$$\therefore \boxed{C \cap \partial E \subset \Gamma}$$

Since $0 \in \partial E$, $u(0) = 0$. Hence, from $\operatorname{Lip}(u) \le 1$, we get $|u(z)| < 1$ $\forall z \in D$. Thus, $\Gamma \subset C$.

Step one: We now prove that either (3) or (4) is true.

Since $|\mu_E|(\mathbb{R}^n \setminus \partial^* E) = 0 \Rightarrow |\mu_E|(\Gamma_+) = |\mu_E|(\Gamma_-) = 0$

We note that Γ_+ and Γ_- are open sets.

Let $\varphi \in C_c^\infty(\Gamma_+)$.

Thus, since E is of finite perimeter, the Generalized Gauss-Green theorem holds:

$$\int_E \nabla\varphi \, dx = \int_{\mathbb{R}^n} \varphi \, d\mu_E .$$

$$\Rightarrow \int_{\mathbb{R}^n} \chi_E \nabla\varphi = \int_{\Gamma_+} \varphi \, d\mu_E \leq \sup|\varphi| \, |\mu_E|(\Gamma_+) = 0$$

$$\therefore \int_{\mathbb{R}^n} \chi_E \nabla\varphi = 0 \qquad \forall \varphi \in C_c^\infty(\Gamma_+)$$

$\Rightarrow \exists\, c_1 \in \mathbb{R}$ s.t. $\chi_E = c_1$ a.e. in Γ_+. See Lemma 7.5 in textbook:

Lemma 7.5 (Vanishing weak gradient). If $u \in L^1_{loc}(\mathbb{R}^n)$, A is open and connected and

$$\int_{\mathbb{R}^n} u \nabla\varphi = 0 \qquad \forall \varphi \in C_c^\infty(A),$$

then $\exists\, c \in \mathbb{R}$ such that $u = c$ a.e. in A

Proof of Lemma 7.5: Say $A = \mathbb{R}^n$. Let $u_\varepsilon = u * \rho_\varepsilon$.

$$\nabla u_\varepsilon(x) = \int_{\mathbb{R}^n} u(y) \nabla\rho_\varepsilon(y-x) \, dy = 0, \quad \forall x \in \mathbb{R}^n.$$

Since u_ε smooth, then $\exists\, c_\varepsilon \in \mathbb{R}$ s.t. $u_\varepsilon = c_\varepsilon$ on $\mathbb{R}^n$. Now, $u_\varepsilon \to u$ in $L^1_{loc}(\mathbb{R}^n)$, and therefore, $\exists\, c \in \mathbb{R}$ s.t. $c_\varepsilon \to c$. Thus, $u = c$ a.e. on $\mathbb{R}^n$

Proceeding in the same way with Γ_- we obtain:

$\exists\, c_2 \in \mathbb{R}$ s.t. $\chi_E = c_2$ a.e. in Γ_-

If $c_1 = c_2 = c$ then $\chi_E = c$ a.e. in C. This would imply: $\int_E \operatorname{div} T = 0$, $\forall T \in C_c^1(C)$, $|T| \leq 1$ $\Rightarrow P(E, C) = 0$, But $P(E;C) = |\mu_E|(C) = 0$ can not be true.

Indeed, $0 \in \partial E \cap C$ and, since $\overline{\partial^* E} = \partial E$, there exists $\tilde{x} \in \Gamma$ such that $\tilde{x} \in \partial^* E$. Hence, since
$$\lim_{r\to 0^+} \frac{|\mu_E|(B(\tilde{x},r))}{\omega_{n-1} r^{n-1}} = 1$$
it follows that, for r small enough, $|\mu_E|(B(\tilde{x},r)) \geq \frac{\omega_n r^{n-1}}{2} > 0$. Hence $|\mu_E|(C) > 0$.

We conclude that $C_1 \neq C_2$ and thus:
$$\begin{cases} \text{either } \chi_E = 1 \text{ a.e. on } \Gamma_+ \\ \text{and } \chi_E = 0 \text{ a.e. on } \Gamma_- \end{cases} \quad \text{or} \quad \begin{array}{l} \chi_E = 0 \text{ a.e. on } \Gamma_- \\ \text{and } \chi_E = 1 \text{ a.e. on } \Gamma_+ . \end{array}$$
which shows that either (3) or (4) holds.

<u>Step two</u>: We now prove (2). We saw before that $C \cap \partial E \subset \Gamma$. We are left to show that $\Gamma \subset C \cap \partial E$, to be able to conclude that $\Gamma = C \cap \partial E$. To see this, we fix $x \in \Gamma$. Consider the cones
$$K_+ = \{(z,t) : t > \mathrm{Lip}(u)|z|\}, \quad K_- = \{(z,t) : t < -\mathrm{Lip}(u)|z|$$
By definition of Lipschitz function we have:
$$C \cap (x+K_+) \subset \Gamma_+ , \quad C \cap (x+K_-) \subset \Gamma_-$$

We have that
$$0 < |E \cap B(x,r)| < \omega_n r^n,$$
because, for example, say that (4) is true, then, if $B(x,r) \subset C$ we have:
$$\frac{|E \cap B(x,r)|}{\omega_n r^n} = \frac{|\Gamma_+ \cap B(x,r)|}{\omega_n r^n} \geq \frac{|(x+K_+) \cap B(x,r)|}{\omega_n r^n} = \frac{|K_+ \cap B(0,1)|}{\omega_n} > 0$$

$$\frac{|B(x,r)\setminus E|}{\omega_n r^n} \geq \frac{|\Gamma_- \cap B(x,r)|}{\omega_n r^n} \geq \frac{|(x+K_-)\cap B(x,r)|}{\omega_n r^n} = \frac{|K_- \cap B(0,1)|}{\omega_n} > 0,$$

and by symmetry, we have the same conclusion if (3) holds instead of (4), that is:

$$0 < |E \cap B(x,r)| < \omega_n r^n, \quad \forall x \in \Gamma, \quad \forall r > 0.$$

Hence (see Lecture 12, Page 12.10):

$$\Gamma \subset \operatorname{spt} \mu_E = \partial E.$$

Step three: We see (5) and (6) we use the following results from the textbook:

Theorem 9.1: (Area of a graph of codimension one). If $u : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function, then for every Lebesgue measurable set G in $\mathbb{R}^{n-1}$,

$$\mathcal{H}^{n-1}(\Gamma(u;G)) = \int_G \sqrt{1 + |\nabla' u(z)|^2}\, dz$$

and

Exercise 10.6 (Tangent space to a graph) If $u : \mathbb{R}^{n-1} \to \mathbb{R}$ is Lipschitz, we define $f : \mathbb{R}^{n-1} \to \mathbb{R}^n$ as $f(z) = (z, u(z))$, $z \in \mathbb{R}^{n-1}$. Then $\Gamma = f(\mathbb{R}^{n-1})$ is locally $\mathcal{H}^{n-1}$ rectifiable and, for a.e. $z \in \mathbb{R}^{n-1}$,

$$T_{f(z)}\Gamma = \nu(z)^\perp, \quad \nu(z) = (-\nabla' u(z), 1).$$

The area functional is defined as:

$$A(u;B) = \int_B \sqrt{1+|\nabla u(x)|^2}\, dx, \quad B \subset \mathbb{R}^n$$

for every Lipschitz function $u: \mathbb{R}^n \to \mathbb{R}$. We say that u is a local minimizer of the area functional in B if for every compact set $K \subset B$ there exists $\varepsilon > 0$ with

$$A(u;B) \leq A(u+\varphi;B), \quad \forall \varphi \in C_c^\infty(B),\ \operatorname{spt}\varphi \subset K,\ \sup|\varphi| \leq \varepsilon$$

We have the following:

Proposition: (Local minimizers of the area functional). Under the assumptions of the Lipschitz graph criterion, if E is further assumed to be a local perimeter minimizer in C, then u is a local minimizer of the area functional in D.

Proof: WLOG we assume that (3) holds in Theorem 1 (Page 25.3). Fix $r \in (0,1)$. We will show:

$$A(u;D) \leq A(u+\varphi;D) \quad \forall \varphi \in C_c^1(D_r),\ \sup|\varphi| < 1-r.$$

$u(0)=0$ and $\mathrm{Lip}(u) \leq 1 \Rightarrow \sup_{D_r} |u+\varphi| < 1,$

Let:

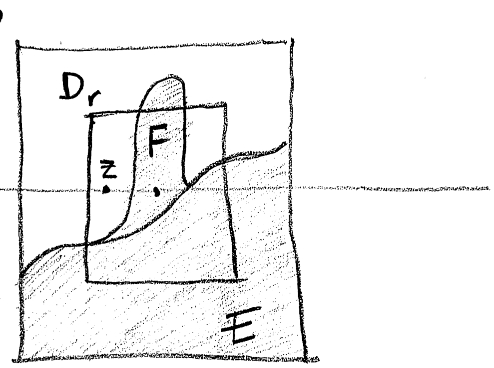
$$F = \{(z,t) : z \in D,\ -1 < t < u(z) + \varphi(z)\},$$

$z \in D_r \Rightarrow$

$|u(z) - u(0)| \leq \mathrm{Lip}(u)|z|$

$\leq |z|$

$\leq r$



$|u(z)| \leq r \Rightarrow$

$|(u+\varphi)(z)| \leq r + 1 - r = 1$

$$\int_D \sqrt{1 + |\nabla' u + \nabla' \varphi|^2}\, dz$$

$$\|$$

$E \Delta F \subset\subset C \quad \Rightarrow \quad P(E;C) \leq P(F;C)$

$$\|$$

$$\mathcal{H}^{n-1}(\partial^* E \cap C)$$

$$\|$$

$$\mathcal{H}^{n-1}(\partial E \cap C)$$

$\|$ ∂E is the graph of u on D

$$\int_D \sqrt{1 + |\nabla' u(z)|^2}\, dz$$

$\therefore\ A(u;D) \leq A(u+\varphi; D), \quad \forall \varphi$ $\square$

We have:

Theorem (Euler–Lagrange equation): $u: \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz local minimizer of $A(u;B) = \int_B \sqrt{1+|\nabla u|^2}\, dx$ in B if and only if:

$$\int_B \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \cdot \nabla \varphi = 0 \quad \forall \varphi \in C_c^\infty(B), \qquad (7)$$

If, moreover $u \in C^2(B)$, then (7) is equivalent to:

$$-\operatorname{div}\left(\frac{\nabla u(x)}{\sqrt{1+|\nabla u|^2}}\right) = 0, \quad \forall x \in B \qquad (8)$$

Remark: Equations (7) and (8) are called, respectively, the weak an strong form of the Euler-Lagrange equation of A. Equation (8) is called the minimal surface equation.

In order to obtain (7) and (8) we apply the classical argument in Calculus of Variations:

With every variation $\varphi \in C_c^\infty(B)$ we define:

$$\Phi : \mathbb{R} \to \mathbb{R}$$

$$\Phi(t) = A(u + t\varphi), \quad t \in \mathbb{R}$$

$$= \int_B \sqrt{1 + |\nabla(u + t\varphi)|^2}\, dx$$

By dominated convergence theorem we see that $\Phi \in C^1(\mathbb{R})$, with:

$$\phi'(0) = \int_B \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \cdot \nabla \varphi \, dx$$

u Lipschitz local minimizer of A in $B \Rightarrow \exists \varepsilon > 0$ s.t $\Phi(0) \leq \Phi(t)$ $|t| < \varepsilon$

Hence: $\Phi'(0) = 0$, which is (7).

Using the divergence theorem:

$$-\int_B \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \cdot \varphi \, dx = 0 \quad \forall \varphi \in C_c^\infty(B)$$

We now use the following fundamental lemma of the Calculus of Variations: 25.12

Exercise 4.14: Let $A \subset \mathbb{R}^n$ open set, ν Radon measure on $\mathbb{R}^n$ with values in $\mathbb{R}^m$.

If
$$\int_{\mathbb{R}^n} \varphi \cdot d\nu = 0, \quad \forall \varphi \in C_c^\infty(A; \mathbb{R}^m)$$
then $|\nu|(A) = 0$. In particular, if $u \in L^1_{loc}(\mathbb{R}^n; \mathbb{R}^m)$ and:
$$\int_{\mathbb{R}^n} (\varphi(x) \cdot u(x))\, dx = 0 \quad \forall \varphi \in C_c^\infty(A; \mathbb{R}^m)$$
then $u = 0$ a.e. on A.

Using this lemma, we have from:
$$-\int_B \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \cdot \varphi\, dx = 0 \quad \forall \varphi \in C_c^\infty(B)$$

that:
$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0. \quad \blacksquare$$