Lesson 26.

Compactness for Minimizers.

25.1

<u>Lemma</u>: If A is open, bounded and convex, and $\{E_i\}_{i=1}^{\infty}$ is a sequence of minimizers of perimeter in A then $\exists \{E_{i_k}\}_{k=1}^{\infty}$ a subsequence such that:

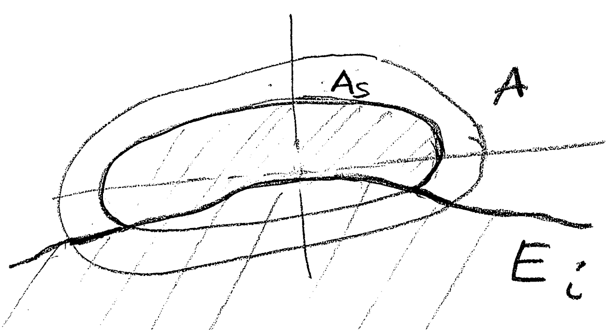
$$E_{i_k} \cap A \longrightarrow E \text{ in } L^1(\mathbb{R}^n)$$

<u>Proof</u>: Recall, that by compactness of sets of finite perimeter we need to show $\exists C$ s.t:

$$P(E_i \cap A) < C \quad \forall i$$

WLOG, $0 \in A$. Let $A_s := sA$, $s < 1$

For a.e. s, $\mathcal{H}^{n-1}(\partial^* E_i \cap \partial A_s) = 0$



Let:
$F_i = E_i \cup A_s$

$$\Rightarrow \quad P(E_i; A) \leq P(F_i; A) \leq P(A_s) + P(E_i; A \setminus A_s)$$

Since $A \setminus A_s \rightarrow \emptyset$ as $s \rightarrow 1$, so:

$$P(E_i; A) \leq P(A)$$

$$\Rightarrow P(E_i \cap A_s) \leq P(E_i; A_s) + P(A_s)$$

$s \rightarrow 1$ gives $P(E_i \cap A) \leq P(E_i; A) + P(A)$
$\leq 2P(A)$ ☐

<u>Theorem</u>: If $\{E_k\}$, E are given by previous Lemma, then

(a) E minimizes perimeter in A

(b) $\mu_{A\cap E_k} \xrightarrow{*} \mu_E$

(c) $|\mu_{A\cap E_k}| \xrightarrow{*} |\mu_E|$

Moreover:

(d) If $\{x_k\}$, $x_k \in \partial E_k \cap A$, $x_k \to x \in A$, then $x \in \partial E \cap A$

(e) $\forall x \in \partial E \cap A$ $\exists x_k$, $x_k \in \partial E_k \cap A$ s.t. $x_k \to x$.

<u>Proof</u>:

(b) follows from the Compactness theorem for sets of finite perimeter (see Lecture 13, Page 13.6).

(c) does not follow from (b) in general, because cancellations can occur.

<u>Step one</u>: We assume $0 \in A$ (otherwise we make a translation). Define:

$$u_j(x) = \inf \{ \lambda > 0 \mid \frac{x}{\lambda} \in A \}$$

Then $\{x : u_j(x) = s\} = \partial A_s$, $A_s := sA$

For every G Borel set:

$$\int_a^b \mathcal{H}^{n-1}(G\cap\partial A_s)\,ds = \int_{G\cap(A_b\setminus A_a)} |\nabla u_j|\,dx$$

$$\leq C\,|G\cap(A_b\setminus A_a)| \;; \quad \text{since } |\nabla u_j| \leq C$$

Step two ; We know from previous Lemma that:

$A\cap E_k \to E$ in $L^1(\mathbb{R}^n)$

$\mu_{A\cap E_k} \xrightarrow{*} \mu_E$

$|\mu_{A\cap E_k}| = P(A\cap E_k) \leq 2P(A).$

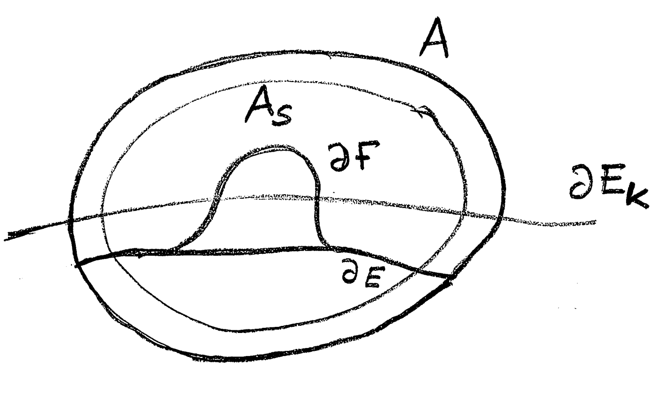
Thus, up to a subsequence,

$|\mu_{A\cap E_k}| \xrightarrow{*} \nu$, for some measure ν.

and $|\mu_E| \leq \nu$

To prove (c) we need to show that $|\mu_E| = \nu$.

In order to prove (a) we construct a comparison sequence of sets:



Let F be such that $E\Delta F \subset\subset A$

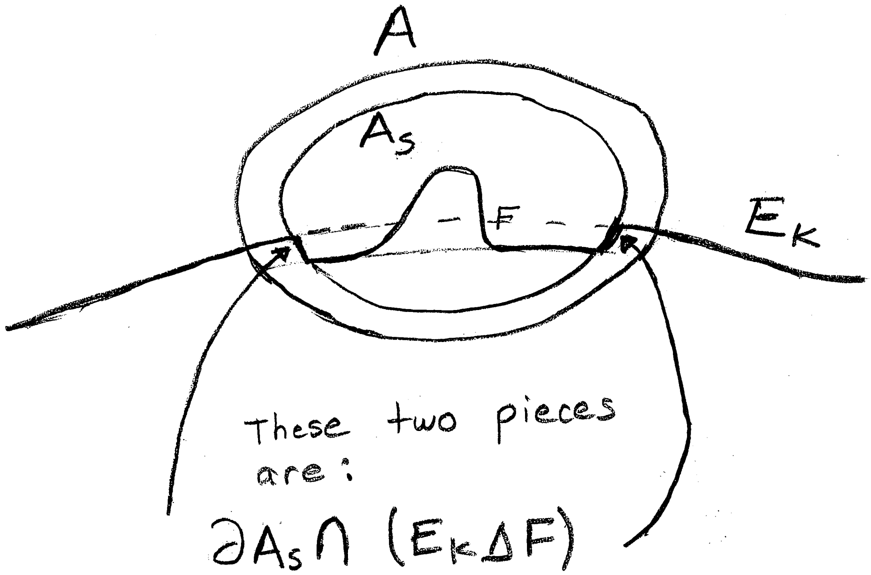
Hence $\exists s_0 < 1$ s.t ;

$E\Delta F \subset\subset A_{s_0}$

Also, for a.e. $s\in(0,1)$ and $\forall k\in\mathbb{N}$;

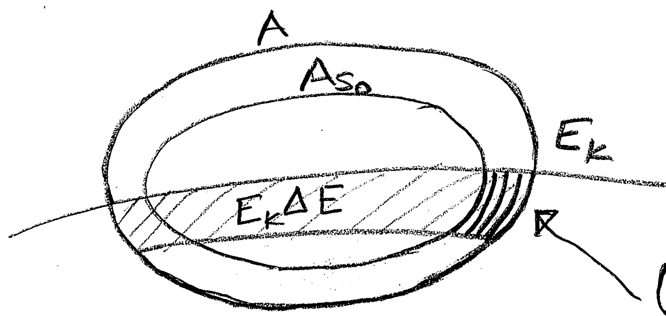
$$\boxed{\mathcal{H}^{n-1}(\partial^* F\cap\partial A_s) = \mathcal{H}^{n-1}(\partial^* E_k\cap\partial A_s) = 0} \quad (*)$$

Comparison set (Connect E_k with F):



We need to pick a "good" s in this picture.

From Step 1, with $G = E_k \Delta F$:



$$\boxed{\int_{s_0}^{1} \mathcal{H}^{n-1}(\partial A_s \cap (E_k \Delta F))\, ds \leq C \left| (A \setminus A_s) \cap (E_k \Delta F) \right|}$$

Hence:

$$0 = \lim_{k \to \infty} \left| (A \setminus A_0) \cap (E_k \Delta F) \right| \geq \frac{1}{C} \liminf_{k \to \infty} \int_{s_0}^{1} \mathcal{H}^{n-1}(\partial A_s \cap (E_k \Delta F))\, ds$$

$$\geq \frac{1}{C} \int_{s_0}^{1} \liminf_{k \to \infty} \mathcal{H}^{n-1}(\partial A_s \cap (E_k \Delta F))\, ds$$

By Fatou's Lemma

Hence:

$$\int_{s_0}^{1} \liminf_{k\to\infty} \mathcal{H}^{n-1}(\partial A_s \cap (E_k \Delta F))\, ds = 0,$$

which implies that, for almost every $s_0 < s < 1$:

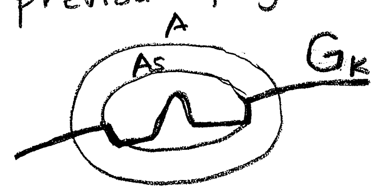
$$\boxed{\liminf_{k\to\infty} \mathcal{H}^{n-1}(\partial A_s \cap (E_k \Delta F)) = 0 \qquad (**)}$$

Fix $s \in (s_0, 1)$ such that $(*)$ and $(**)$ hold;

Let

$$\boxed{G_k := (F \cap A_s) \cup (E_k \setminus A_s)}$$

See first figure in previous page:



To show (a), that E is a minimizer, we compute:

$$|\mu_E|(A_s) = P(E; A_s)$$

$$\leq \nu(A_s) \quad ; \text{ since } |\mu_E| \leq \nu$$

$$\leq \liminf_{k\to\infty} |\mu_{E_k \cap A}|(A_s) \quad ; \text{ since } |\mu_{E_k \cap A}| \xrightarrow{*} \nu$$

But; since each E_k is a minimizer:

$$\boxed{P(E_k; A) \leq P(G_k; A)} = P(G_k; A_s) + P(G_k; A \setminus A_s) + \mathcal{H}^{n-1}(\partial A_s \cap (E_k \Delta F))$$

$$P(E_k; A) = P(E_k; A_s) + P(E_k; A \setminus A_s) + \underbrace{\mathcal{H}^{n-1}(\partial A_s \cap \partial^* E_k)}_{=0 \text{ by } (*)}$$

$$P(G_k; A_s) + P(G_k; A \setminus A_s) + \mathcal{H}^{n-1}(\partial A_s \cap (E_k \Delta F)) = P(F; A_s) + P(E_k; A \setminus A_s) + \mathcal{H}^{n-1}(\partial A_s \cap (E_k \Delta F))$$

Since $P(E_k; A\setminus A_s)$ can be cancelled in previous inequality, we obtained:

$$\boxed{P(E_k; A_s) \leq P(F; A_s) + \mathcal{H}^{n-1}(\partial A_s \cap (E_k \Delta F))} \quad (***)$$

Hence:

$$P(E;A) = P(E; A_s) + P(E; A\setminus A_s)$$
$$\leq \left[\liminf_{k\to\infty} |\mu_{E_k \cap A}|(A_s)\right] + P(F; A\setminus A_s)$$
$$= \left[\liminf_{k\to\infty} P(E_k; A_s)\right] + P(F; A\setminus A_s)$$
$$\leq P(F; A_s) + \left[\liminf_{k\to\infty} \mathcal{H}^{n-1}(\partial A_s \cap (E_k \Delta F))\right] + P(F; A\setminus A_s) \text{ ; by } (***)$$
$$= P(F; A_s) + P(F; A\setminus A_s) \text{; by } (**)$$
$$= P(F; A)$$

$\therefore$ $P(E;A) \leq P(F;A) \implies$ (a) holds

To prove (c); aply previous inequalities to $F = E$;

$$P(E; A_s) \leq \nu(A_s) \leq P(E; A_s) = |\mu_E|(A_s)$$

Hence $\nu(A) \leq |\mu_E|(A)$, and since we knew that $|\mu_E| \leq \nu$, we conclude that $|\mu_E|(A) = \nu(A)$

But $|\mu_E| \leq \nu$ plus $|\mu_E|(A) = \nu(A)$ imply that

$$|\mu_E| = \nu$$

<u>Proof of (d)</u>: Here we use that:

$$|\mu_{E_k \cap A}| \xrightarrow{*} |\mu_E|$$

and hence:

$$P(E; B(x,r)) \leq \liminf_{k\to\infty} P(E_k, B_r(x)), \qquad (1)$$

for every $B(x,r) \subset\subset A$, and

$$\limsup_{k\to\infty} P(E_k; \overline{B(x,r)}) \leq P(E; \overline{B(x,r)}) \qquad (2)$$

for every $B(x,r) \subset\subset A$. Actually (1) and (2) hold, by weak convergence, for every open set $A' \subset A$ and for every closed set $K \subset A$, respectively.

We have $x_k \in A \cap \partial E_k$, $x_k \to x \in A$.

Let $r > 0$ s.t. $B(x, 2r) \subset\subset A$

For k large enough, $B(x_k, r) \subset B(x, 2r)$.

By monotonicity:

$$\boxed{\omega_{n-1} r^{n-1} \leq \mathcal{H}^{n-1}(\partial E_k \cap B(x_k, r))}$$

Indeed, recall that since E_k is a minimizer in A, then $\mathcal{H}^{n-1}(\partial E_k \setminus \partial^* E_k) = 0$ and $\lim_{r\to 0} \frac{\mathcal{H}^{n-1}(\partial E_k \cap B(x,r))}{\omega_{n-1} r^{n-1}} \geq 1$ $\forall x \in \partial E_k \cap A$; see Lecture 23, page 23.10.

Therefore:

$$\omega_{n-1} r^{n-1} \leq \mathcal{H}^{n-1}(\partial E_k \cap B(x_k, r)) \leq \mathcal{H}^{n-1}(\partial E_k \cap \overline{B(x, 2r)})$$

letting $k \to \infty$ and using (2):

$$\limsup_{k \to \infty} \mathcal{H}^{n-1}(\partial E_k \cap \overline{B(x, 2r)}) \leq \mathcal{H}^{n-1}(\partial E \cap \overline{B(x, 2r)}).$$

$$\therefore \quad \omega_{n-1} r^{n-1} \leq \mathcal{H}^{n-1}(\partial E \cap \overline{B(x, 2r)})$$

$$\| \quad \mathcal{H}^{n-1}(\partial^* E \cap \overline{B(x, 2r)}); \text{ since } E \text{ is a minimizer}$$

We can write this as:

$$|\mu_E| (B(x, r)) > 0 \quad \forall r > 0,$$

which means that $x \in \operatorname{spt} \mu_E$, but since $\operatorname{spt} \mu_E = \partial E$, we conclude that $x \in \partial E$

Proof of (e): We proceed by contradiction, then $\exists x \in \partial E \cap A$, $r > 0$ s.t. $B(x, r) \subset\subset A$ and

$$B(x, r) \cap \partial E_k = \emptyset$$

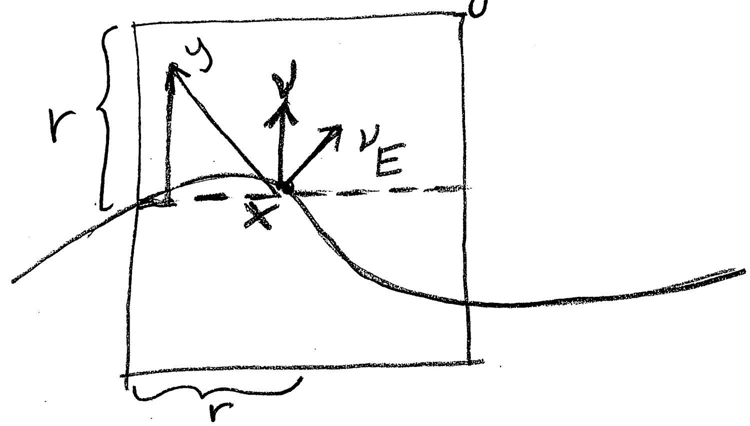
Using (1) and the monotonicity, we get the following contradiction:

$$\omega_{n-1} r^{n-1} \leq P(E; B(x, r)) \leq \liminf_{k \to \infty} P(E_k; B(x, r)) = 0 \quad \blacksquare$$

The Excess

We now introduce the concept of excess, which is a Key concept in the regularity theory for minimizers.

Recall our notations, given in Lesson 25, page 25.4:



$x \in \partial E$

ν any unit vector.

$$C(x,r,\nu) = \{ y : |(y-x)\cdot\nu| < r,\ |y-x-((y-x)\cdot\nu)\nu| < r \}$$

open cylinder

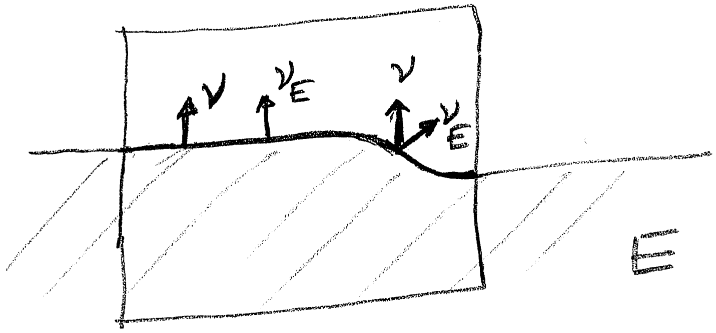
The <u>cylindrical excess</u> of E at $x \in \partial E$, at the scale $r > 0$, with respect to the direction $\nu \in S^{n-1}$, is defined as:

$$e(E,x,r,\nu) := \frac{1}{r^{n-1}} \int_{C(x,r,\nu)\cap\partial^* E} \frac{|\nu_E(x) - \nu|^2}{2}\, d\mathcal{H}^{n-1}(y)$$

$$= \frac{1}{r^{n-1}} \int_{C(x,r,\nu)\cap\partial^* E} (1 - \nu_E\cdot\nu)\, d\mathcal{H}^{n-1}(y)$$

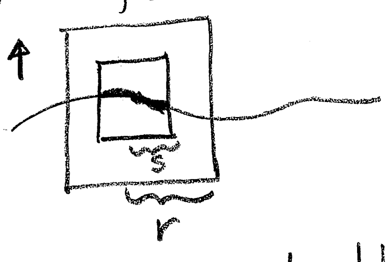
The spherical excess of E at the point $x \in \partial E$ and at the scale $r>0$ is defined as:

$$e(E,x,r) = \min_{\nu \in S^{n-1}} \int_{B(x,r)\cap \partial^* E} \frac{|\nu_E(y)-\nu|^2}{2}\, d\mathcal{H}^{n-1}(y)$$



Notes:

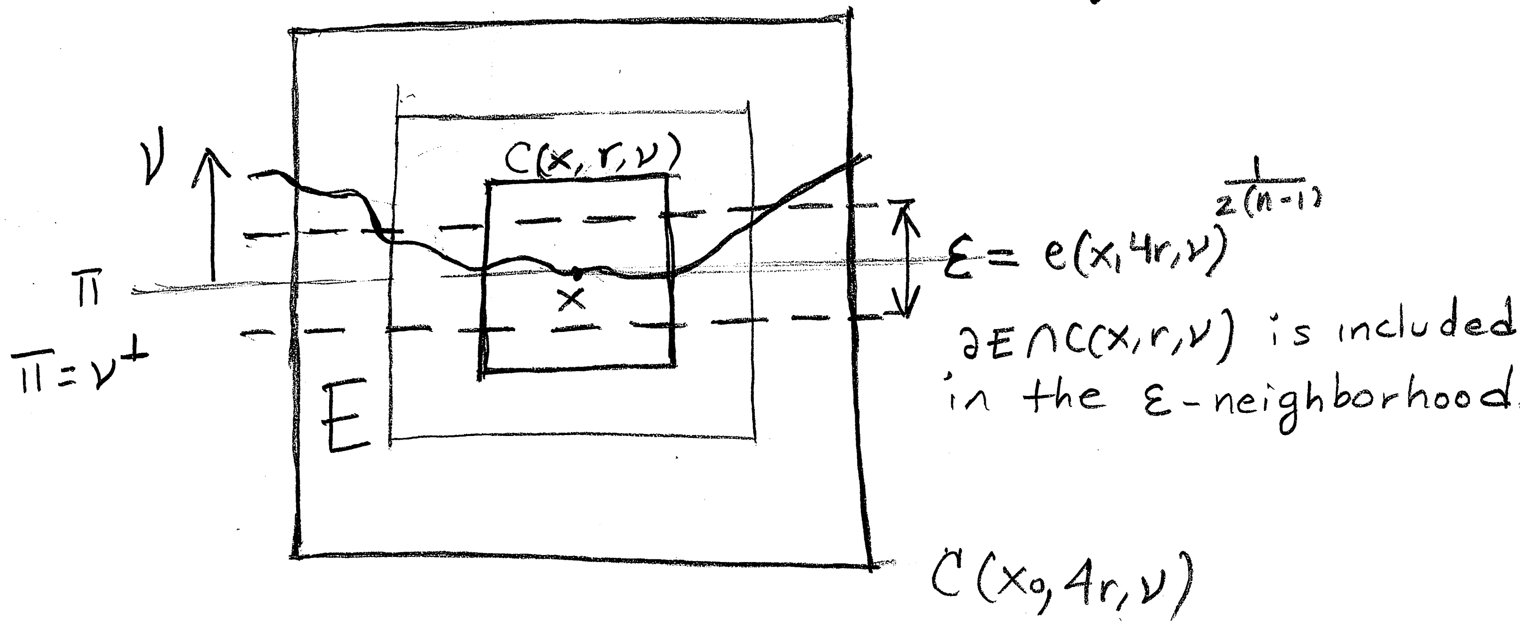
- We will later prove that the smallness of $e(E,x,r,\nu)$ at some $x\in\partial E$ actually forces $C(x,s,\nu)\cap\partial E$ (for some $s<r$) to agree with the graph (with respect to the direction ν) of a $C^{1,\gamma}$ function.



This will be proved through a long series of intermediate results, which include the so-called height bound Theorem, which says that if E is a minimizer in $C(x,4r,\nu)$, $x\in\partial E$, and $e(x,4r,\nu)$ is suitably small, then the uniform distance of

$C(x,r,\nu)\cap\partial E$ from the hyperplane passing through x and orthogonal to ν is bounded from above by $e(x,4r,\nu)^{\frac{1}{2(n-1)}}$

26.11



Basic properties of the excess

Proposition: E has locally finite perimeter, $\operatorname{spt}\mu_E=\partial E$, $r>0$, $\nu\in S^{n-1}$. Then

(i) With $E_{x,r}=\frac{E-x}{r}$, then

$$e(E,x,r,\nu)=e(E_{x,r},0,1,\nu)$$

$$e(E,x,r)=e(E_{x,r},0,1)$$

(ii) $e(E,x,r,\nu)=0$ implies being a half-space:

$$E\cap C(x,r,\nu)=\{y\in C(x,r,\nu):(y-x)\cdot\nu\le 0\}$$

(iii) If $x \in \partial^* E$ then:

$$\lim_{r \to 0^+} e(E, x, r) = 0$$

Hence, $\forall \varepsilon > 0$, $\exists r_0 > 0$ and $\nu \in S^{n-1}$ s.t. $\underset{r \le r_0}{e(E, x, r, \nu)} \le \varepsilon$.

<u>Proof</u>:

For (ii) we can argue as in the proof of the De Giorgi's theorem (see Lecture 18, Lemma 3, page 18.9).

(i) clearly follows from the definition of excess and:

$$\mu_{E_{x,r}} = \frac{(\Phi_{x,r})_{\#} \mu_E}{r^{n-1}} \quad ; \quad \text{see Lemma 1, Lesson 18, Page 18.7}$$

which implies:

$$|\mu_{E_{x,r}}|(B) = \frac{1}{r^{n-1}} |\mu_E| (B(x,r))$$

For (iii):

$$\frac{|\nu_E - \nu|^2}{2} = 1 - \nu_E \cdot \nu$$

$$e(E, x, r) = \frac{1}{r^{n-1}} \inf_{\nu} \int_{\partial^* E \cap B(x,r)} (1 - \nu_E \cdot \nu)\, d\mathcal{H}^{n-1} \quad , \quad x \in \partial^* E$$

$$= \inf_{\nu} \frac{|\mu_E|(B(x,r)) - \mu_E(B(x,r)) \cdot \nu}{r^{n-1}}$$

$$= \frac{|\mu_E|(B(x,r))}{r^{n-1}} \inf_{\nu} \left(1 - \frac{\mu_E(B(x,r))}{|\mu_E|(B(x,r))} \cdot \nu\right)$$

$$x \in \partial^* E \implies \frac{\mu_E(B(x,r))}{|\mu_E|(B(x,r))} \longrightarrow \nu_E(x) \quad \text{as } r \to 0.$$

So, <u>if $\nu = \nu_E$</u>, this goes to 0 as $r \to 0$.