Lesson 27.

More properties of the Excess:

Proposition (Lower semicontinuity of the excess): $A \subset \mathbb{R}^n$ open set, bounded, convex. Let $\{E_k\}$ be a sequence of minimizers for perimeter in A, such that

$$A \cap E_k \to E \text{ in } L^1(\mathbb{R}^n).$$

Then, for every cylinder $C(x,r,\nu) \subset\subset A$ we have:

$$e(E,x,r,\nu) \leq \liminf_{k\to\infty} e(E_k,x,r,\nu).$$

In fact, if $C(x,r,\nu)$ is such that:

$$\mathcal{H}^{n-1}(\partial^* E \cap \partial C(x,r,\nu)) = 0,$$

then we have exactly:

$$e(E,x,r,\nu) = \lim_{k\to\infty} e(E_k,x,r,\nu).$$

Proof: In the previos Lesson we showed that:

- $\nu_{E_k\cap A}\mathcal{H}^{n-1} \llcorner \partial(E_k \cap A) = \mu_{E_k\cap A} \xrightarrow{*} \mu_E = \nu_E \mathcal{H}^{n-1} \llcorner \partial E$
- $\mathcal{H}^{n-1} \llcorner \partial(E_k\cap A) = |\mu_{E_k\cap A}| \xrightarrow{*} |\mu_E| = \mathcal{H}^{n-1} \llcorner \partial E$

We note that at this point we can use:

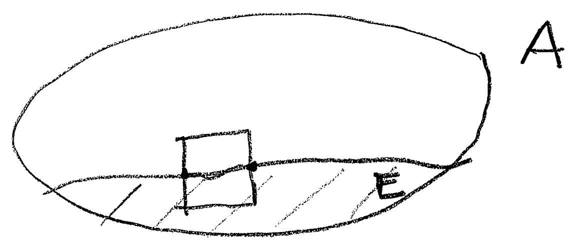
$$\partial \quad \text{or} \quad \partial^*,$$

because, recall that for any minimizer F:

$$\mathcal{H}^{n-1}(\partial F \setminus \partial^* F) = 0$$

Now, if we have that:

$$\mathcal{H}^{n-1}(\partial E \cap \partial C(x,r,\nu)) = 0$$



Since $|\mu_E|(\partial C(x,r,\nu)) = \mathcal{H}^{n-1}(\partial E \cap \partial C(x,r,\nu)) = 0$, then recall that, from standard properties of weak convergence of measures, we have:

$$\mu_{E_k \cap A}(C(x,r,\nu)) \to \mu_E(C(x,r,\nu))$$

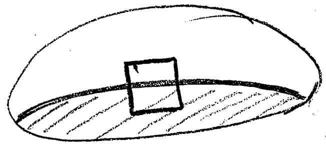
$$|\mu_{E_k \cap A}|(C(x,r,\nu)) \to |\mu_E|(C(x,r,\nu)).$$

Now:

$$\boxed{e(E_k,x,r,\nu) = \frac{1}{r^{n-1}} \int_{\partial^* E_k \cap C(x,r,\nu)} (1 - \nu_E \cdot \nu)\, d\mathcal{H}^{n-1} = \frac{|\mu_{E_k}|(C(x,r,\nu)) - \mu_{E_k}(C(x,r,\nu)) \cdot \nu}{r^{n-1}}} \quad (1)$$

Notice that, since $C(x,r,\nu) \subset A$, we have that

(2) $$\boxed{\begin{aligned} |\mu_{E_k}|(C(x,r,\nu)) &= |\mu_{E_k \cap A}|(C(x,r,\nu)) \\ \mu_{E_k}(C(x,r,\nu)) &= \mu_{E_k \cap A}(C(x,r,\nu)) \end{aligned}}$$

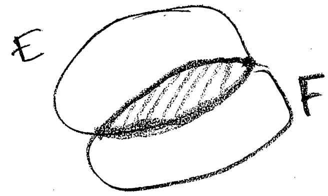


Indeed, recall the formulas in Lecture 21, Page 21.1, where for example:

$$\underset{\underset{|\mu_{E \cap F}|(G)}{\shortparallel}}{P(E \cap F; G)} = P(E; F^{(1)} \cap G) + P(F; E^{(1)} \cap G) + \mathcal{H}^{n-1}(\{\nu_E = \nu_F\} \cap G)$$

$$\Downarrow$$

$$\{\nu_E = \nu_F\} \subset \partial^* E \cap \partial^* F)$$



Hence, from (1) and (2);

$$e(E_k, x, r, \nu) = \frac{|\mu_{E_k \cap A}|(C(x,r,\nu)) - \mu_{E_k \cap A}(C(x,r,\nu)) \cdot \nu}{r^{n-1}}$$

$$\Big\downarrow k \to \infty$$

$$\frac{|\mu_E|(C(x,r,\nu)) - \mu_{E \cap A}(C(x,r,\nu)) \cdot \nu}{r^{n-1}}$$

$$\shortparallel$$

$$e(E, x, r, \nu)$$

$$\therefore \boxed{e(E_k, x, r, \nu) \longrightarrow e(E, x, r, \nu)} \quad (3)$$

We fix now any $r>0$ s.t.:

$$C(x,r,\nu) \subset\subset A.$$

We can choose a sequence $\{r_j\}$, $r_j \uparrow r$ such that:

$$\boxed{\mathcal{H}^{n-1}(\partial E \cap \partial C(x,r_j,\nu)) = 0} \quad (4)$$

Note that since $r_j < r_{j+1} < r$:

$$e(E_k,x,r_j,\nu) = \frac{1}{r_j^{n-1}} \int_{\partial^* E_k \cap C(x,r_j,\nu)} \frac{|\nu_E-\nu|^2}{2} \leq \frac{1}{r_j^{n-1}} \int_{\partial^* E_k \cap (C(x,r,\nu))} \frac{|\nu_E-\nu|^2}{2}$$

$$= \left(\frac{r}{r_j}\right)^{n-1} \cdot \frac{1}{r^{n-1}} \int_{\partial^* E_k \cap (C(x,r,\nu))} \frac{|\nu_E-\nu|^2}{2}$$

$$\therefore \; e(E_k,x,r_j,\nu) \leq \left(\frac{r}{r_j}\right)^{n-1} e(E_k,x,r,\nu)$$

Therefore:

$$e(E,x,r_j,\nu) = \lim_{k\to\infty} e(E_k,x,r_j,\nu) \text{; from (3) and (4)}$$

$$\leq \liminf_{k\to\infty} \left(\frac{r}{r_j}\right)^{n-1} e(E_k,x,r,\nu)$$

$$= \left(\frac{r}{r_j}\right)^{n-1} \liminf_{k\to\infty} e(E_k,x,r,\nu)$$

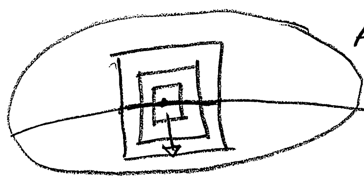
We now show, directly from the definition of the excess, that:

$r \mapsto e(E,x,r,\nu)$ is continuous from the left on $(0,\infty)$, that is,

$$e(E,x,r,\nu) = \lim_{s \to r^-} e(E,x,s,\nu)$$

Indeed, for any sequence $s_j \to r$, $s_j < s_{j+1} < r$:

$$\int_{C(x,s_j,\nu)\cap\partial^* E} \frac{|\nu_E(y)-\nu|^2}{2} \xrightarrow[j\to\infty]{} \int_{C(x,r,\nu)\cap\partial^* E} \frac{|\nu_E(y)-\nu|^2}{2}$$



A

"convergence from inside".

Thus, for any sequence $s_j \to r$, $s_j < s_{j+1} < r$ we have that $e(E,x,s_j,\nu) \to e(E,x,r,\nu)$. In the general case, given any sequence $\{s_j\}$, $s_j \to r$, $s_j < r$, we can extract a subsequence s_{j_i}, $s_{j_i} < s_{j_{i+1}} < r$ and thus $e(E,x,s_{j_i},\nu) \to e(E,x,r,\nu)$ as $i \to \infty$.

Note that this property is enough to conclude that

$$e(E,x,r,\nu) = \lim_{s\to r^-} e(E,x,s,\nu).$$

Going back to our inequality:

$$e(E,x,r_j,\nu) \le \left(\frac{r}{r_j}\right)^{n-1} \liminf_{k\to\infty} e(E_k,x,r,\nu)$$

Letting now $r_j \to r$, using that $r \mapsto e(E,x,r,\nu)$ is left continuous, we conclude the lower semicontinuity:

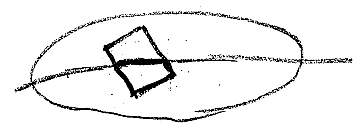
$$e(E,x,r,\nu) \le \liminf_{k\to\infty} e(E_k,x,r,\nu). \quad \blacksquare$$

Remark: If E minimizes perimeter in A, the following are true:

(a) $P(E; C(x,r,\nu)) \geq P(E; B(x,r))$; since $B(x,r) \subset C(x,r,\nu)$

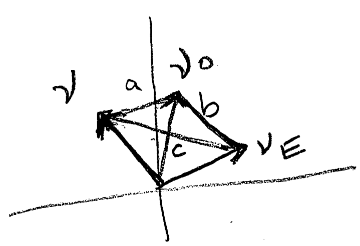
$\geq \omega_{n-1} r^{n-1}$; by Monotonicity.

(b) $P(E; C(x,r,\nu)) \leq P(C(x,r,\nu))$; because E is a minimizer



$= P(C(0,1,\nu))\, r^{n-1}$

(c) Since $|\nu_E - \nu|^2 \leq 2(|\nu_E - \nu_0|^2 + |\nu_0 - \nu|^2)$

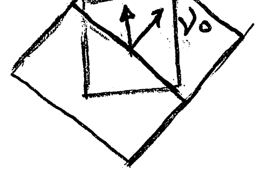


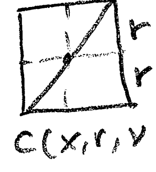
$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos\theta \\ &\leq a^2 + b^2 + 2|a||b| \\ &\leq a^2 + b^2 + 2\left(\frac{a^2}{2} + \frac{b^2}{2}\right) = 2a^2 + 2b^2 \end{aligned}$$

Hence

$$e(E,x,r,\nu) = \frac{1}{r^{n-1}} \int_{\partial^* E \cap C(x,r,\nu)} \frac{1}{2}|\nu_E - \nu|^2 \, d\mathcal{H}^{n-1} \leq \frac{1}{r^{n-1}} \int_{\partial^* E \cap C(x,r,\nu)} \frac{1}{2}|\nu_E - \nu_0|^2 \, d\mathcal{H}^{n-1} + \frac{1}{r^{n-1}} \int_{\partial^* E \cap C(x,r,\nu)} \frac{1}{2}|\nu_0 - \nu|^2 \, d\mathcal{H}^{n-1}$$

Since $C(x,r,\nu) \subset C(x,\sqrt{2}r,\nu_0)$ it follows that:





$C(x,r,\nu)$

$$d = \sqrt{4r^2 + 4r^2} = 2(\sqrt{2}r).$$

$\Longrightarrow$

$$e(E,x,r,\nu) \le \frac{1}{r^{n-1}} \cdot \frac{(\sqrt{2}r)^{n-1}}{(\sqrt{2}r)^{n-1}} \int_{\partial^* E \cap C(x,\sqrt{2}r,\nu_0)} \frac{1}{2}|\nu_E - \nu_0|^2 + \frac{|\nu_0-\nu|^2}{2r^{n-1}} P(E; C(x,r,\nu))$$

$$\le (\sqrt{2})^{n-1} \cdot \frac{1}{(\sqrt{2}r)^{n-1}} \int_{\partial^* E \cap C(x,\sqrt{2}r,\nu_0)} \frac{1}{2}|\nu_E - \nu_0|^2 + \frac{|\nu_0-\nu|^2}{2r^{n-1}} P(C(0,1,\nu))r^{n-1};$$

by (b)

$$= (\sqrt{2})^{n-1} e(E,x,\sqrt{2}r,\nu_0) + \frac{P(C(0,1,\nu))}{2} |\nu_0 - \nu|^2$$

Hence, we have proved that:

$$\boxed{e(E,x,r,\nu) \le c(n)\left(e(E,x,\sqrt{2}r,\nu_0) + |\nu_0-\nu|^2\right)}$$

(d) Since:

$$\frac{|\nu_E+\nu|^2}{2} + \frac{|\nu_E-\nu|^2}{2} = 2 \quad \text{we have:}$$

$$e(E,x,r,\nu) + e(E,x,r,-\nu) = \frac{2}{r^{n-1}} P(E; C(x,r,\nu))$$

$$\ge \frac{2\,\omega_{n-1} r^{n-1}}{r^{n-1}} \quad ; \text{ by (a)}$$

$$= 2\omega_{n-1}$$

Hence:

$$\boxed{e(E,x,r,\nu) + e(E,x,r,-\nu) \ge 2\,\omega_{n-1}}$$

In order to prove the "height bound theorem", we need to show first two intermediate Lemmas.

Lemma (Small-excess position): For every $n \geq 2$ and $t_0 \in (0,1)$, there exists a positive constant $\omega(n, t_0)$ with the following property:
If E is a perimeter minimizer in $C_2 = C(0,2)$, $0 \in \partial E$, and:

$$e_n(2) \leq \omega(n, t_0)$$

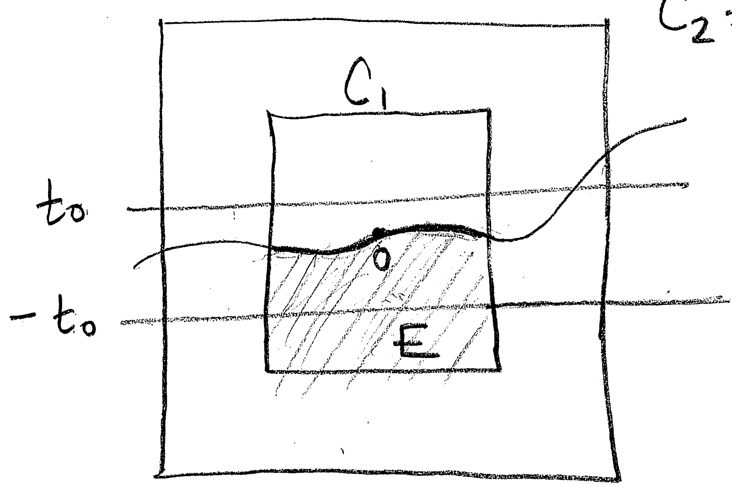
(where we have set, $e_n(s) := e(E, 0, s, e_n)$, $s > 0$), then

(i) $\mathcal{H}^{n-1}(C_1 \cap \partial E \cap \{t_0 < |x_n| < 1\}) = 0$

(ii) $|C_1 \cap E \cap \{x_n > t_0\}| = 0$

(iii) $|(C_1 \setminus E) \cap \{x_n < -t_0\}| = 0$.

$C_2 = C(0,2) = C(0,2,e_n)$



Proof: We proceed by contradiction.

Then $\exists\, t_0 \in (0,1)$, $\exists \{E_k\}$ a sequence of minimizers in C_2, $0 \in \partial E_k$, $e(E_k, 0, 2, e_n) \to 0$ but one of the three consequences is false.

We assume that (i) fails. Hence

$$\mathcal{H}^{n-1}(C_1 \cap \partial E_k \cap \{t_0 < x_n < 1\}) > 0.$$

Therefore, for each k, $\exists\, x_k \in \partial E_k \cap C_1$ with:

$$t_0 < x_k \cdot e_n < 1$$

From the compactness theorem for minimizer proved in a previous lecture, up to a subsequence, still denoted as $\{E_k\}$, we have that:

$$E_k \cap C_2 \longrightarrow F \text{ in } L^1$$
$$x_k \longrightarrow \bar{x}.$$

But we proved in Lecture 26 that $x_k \to \bar{x}$ implies:

$$\bar{x} \in \partial F$$

Moreover, $\bar{x} \in \bar{C}_1$ and $\bar{x} \cdot e_n \geq t_0$; since $t_0 < x_k \cdot e_n < 1$.

Now, by the lower semicontinuity of the excess we compute:

$$e(F, 0, 2, e_n) \leq \liminf_{k \to \infty} e(E_k, 0, 2, e_n) = 0,$$

which implies that $F \cap C_2 = \{x_n \leq 0\}$, which contradicts $\bar{x} \in \partial F$, $\bar{x} \cdot e_n \geq t_0$. If either (ii) or (iii) fails we proceed in a similar way to get a contradiction. ∎

We now prove:

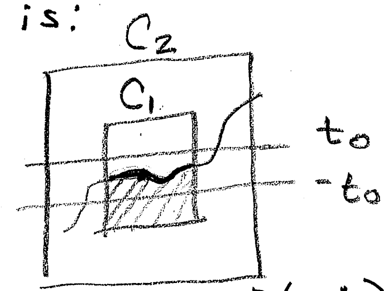
Lemma (Excess measure);

$E \subset \mathbb{R}^n$ set of locally finite perimeter, $0 \in \partial E$. Assume that the three properties (i), (ii), (iii) from previous Lemma hold for some $t_0 \in (0,1)$, that is:

(i) $|qx| < t_0$, $\forall x \in C_1 \cap \partial E$

(ii) $|\{x \in C_1 \cap E : qx > t_0\}| = 0$

(iii) $|\{x \in C_1 \setminus E : qx < -t_0\}| = 0$



$D_1 = D(0,1)$
$C_1 = C(0,r)$

then, with $M := C_1 \cap \partial^* E$, we have:

- $\mathcal{H}^{n-1}(G) \le \mathcal{H}^{n-1}(M \cap p^{-1}(G))$, $\mathcal{H}^{n-1}(G) = \displaystyle\int_{M \cap p^{-1}(G)} (\nu_E \cdot e_n)\, d\mathcal{H}^{n-1}$ (5)

- $\displaystyle\int_{D_1} \varphi = \int_M \varphi(px)(\nu_E(x)\cdot e_n)\, d\mathcal{H}^{n-1}(x)$, $\displaystyle\int_{E_t \cap D_1} \varphi = \int_{M \cap \{qx > t\}} \varphi(px)(\nu_E(x)\cdot e_n)\, d\mathcal{H}^{n-1}(x)$ (6)

for every $G \subset D_1$ Borel, $\varphi \in C_c(D)$, $t \in (-1,1)$, $E_t := \{z \in \mathbb{R}^{n-1} : (z,t) \in E\}$.

Moreover, the set function

$$\zeta(G) = P(E; C_1 \cap p^{-1}(G)) - \mathcal{H}^{n-1}(G)$$
$$= \mathcal{H}^{n-1}(M \cap p^{-1}(G)) - \mathcal{H}^{n-1}(G), \quad G \subset \mathbb{R}^{n-1},$$

defines a Radon measure on $\mathbb{R}^{n-1}$, concentrated on D_1. The Radon measure ζ is called the excess measure of E over D_1 since:

$$\zeta(D_1) = \mathcal{H}^{n-1}(M \cap p^{-1}(D_1)) - \mathcal{H}^{n-1}(D_1)$$
$$= \mathcal{H}^{n-1}(C_1 \cap \partial^* E) - \mathcal{H}^{n-1}(D_1) = \int_{C_1 \cap \partial^* E} (1 - \nu_E(x)\cdot e_n)\, d\mathcal{H}^{n-1} = \underset{e_n(1)}{\underset{\parallel}{e(E, 0, 1, e_n)}}$$

Remark: The lower bound in (5) implies that $C_1 \cap \partial^* E$ "leaves no holes" over D_1.

Since ζ is a positive measure, and $G \subset D_1$, then

$$\zeta(G) \leq \zeta(D_1).$$

Since $\zeta(D_1) = e_n(1)$ we obtain:

$$\mathcal{H}^{n-1}(G) \leq \mathcal{H}^{n-1}(C_1 \cap \partial^* E \cap p^{-1}(G) \leq \mathcal{H}^{n-1}(G) + e_n(1)$$

If $e_n(1)$ is very small then:

$$\mathcal{H}^{n-1}(C_1 \cap \partial^* E \cap p^{-1}(G) \sim \mathcal{H}^{n-1}(G),$$

that is, $C_1 \cap \partial^* E$ is "almost flat".

Proof of the Excess measure Lemma:

(6) $\Rightarrow$ (5) by approximation arguments

Also, we prove (6) for $\varphi \in C_c^1(D_1)$; since again by an approximation argument, it holds for $\varphi \in C_c(D_1)$. By foliation of Radon measures, we have:

$$\mathcal{H}^{n-1}(\partial^* E \cap (\partial D_r \times \mathbb{R})) = 0 \quad \text{a.e. } r \in (0,1). \qquad (7)$$

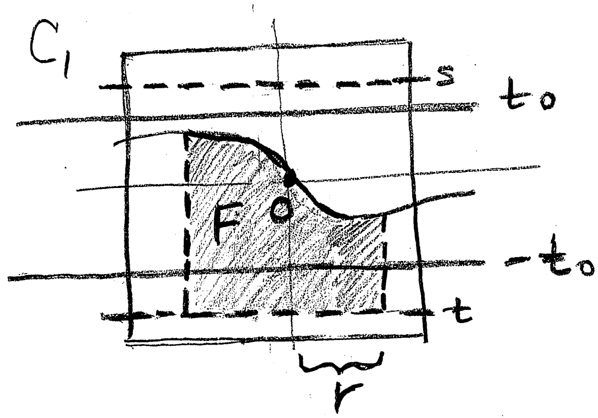
From (ii) and (iii), using Fubini's Theorem:

$$\mathcal{H}^{n-1}(E \cap (D_1 \times \{s\})) = 0 \quad \text{for a.e. } s \in (t_0, 1) \qquad (8)$$

$$\mathcal{H}^{n-1}(E \cap (D_1 \times \{t\})) = \mathcal{H}^{n-1}(D_1) \text{ for a.e. } t \in (-1, t_0) \qquad (9)$$

We let r, s for which (7) and (8) holds. Given $t \in (-1, s)$, we define the set of finite perimeter as:

$$\boxed{F = E \cap (D_r \times (t,s))}$$



$$\mu_F = \mu_E \llcorner (D_r \times (t,s)) + \mu_{D_r \times (t,s)} \llcorner E$$

$\nu(x) = \dfrac{px}{|px|}$ outer normal to $D_r \times \mathbb{R}$ at $x \in \partial D_r \times \mathbb{R}$

Then, (see Exercise 16.4):

$$\mu_{D_r \times (t,s)} = e_n \mathcal{H}^{n-1} \llcorner (D_r \times \{s\}) + \nu\, \mathcal{H}^{n-1} \llcorner (\partial D_r \times (t,s)) - e_n \mathcal{H}^{n-1} \llcorner (D_r \times \{t\})$$

$\Rightarrow$

$$\boxed{e_n \cdot \mu_F = (e_n \cdot \nu_E)\, \mathcal{H}^{n-1} \llcorner (\partial^* E \cap (D_r \times (t,s))) - \mathcal{H}^{n-1} \llcorner (E \cap (D_r \times \{t\})). \quad (*)}$$

Let $\varphi \in C_c^1(D_1)$. Define $T \in C^1(\mathbb{R}^n; \mathbb{R}^n)$:

$$T(x) = \varphi(px)\, e_n, \quad x \in \mathbb{R}^n.$$
$$= (0, 0, \dots, \varphi(px))$$

$\Rightarrow \operatorname{div} T = \dfrac{\partial}{\partial x_n} \varphi(px) = 0$; since $\varphi = \varphi(x_1, \dots, x_{n-1})$

The Generalized Gauss-Green Theorem yields:

$$0 = \int_F \operatorname{div} T = \int_{\mathbb{R}^n} T \cdot d\mu_F$$

$$= \int_{\mathbb{R}^n} \varphi(px)\, e_n \cdot d\mu_F$$

$$= \int_{\partial^* E \cap (D_r \times (t,s))} \varphi(px)\, e_n \cdot \nu_E \, d\mathcal{H}^{n-1} - \int_{E \cap (D_r \times \{t\})} \varphi(px)\, d\mathcal{H}^{n-1}$$

$$\therefore \boxed{\int_{E \cap (D_r \times \{t\})} \varphi(px)\, d\mathcal{H}^{n-1} = \int_{\partial^* E \cap (D_r \times (t,s))} \varphi(px)\, e_n \cdot \nu_E \, d\mathcal{H}^{n-1}} \quad (**)$$

We first let $r \to 1^-$ and then $s \to 1^-$ to get (6):

$$\int_{E_t \cap D_1} \varphi = \int_{E \cap (D_1 \times \{t\})} \varphi(p(x))\, d\mathcal{H}^{n-1} \quad ; \text{ area formula}$$

$$= \int_{\partial^* E \cap (D_1 \times (t,1))} \varphi(px)\, e_n \cdot \nu_E \, d\mathcal{H}^{n-1} \quad ; \text{ above } (**) \text{ with } r \to 1,\ s \to 1.$$

$$= \int_{M \cap \{qx > t\}} \varphi(px)\, (e_n \cdot \nu_E(x))\, d\mathcal{H}^{n-1} ; \text{ since } M \cap \{qx > t\} = \partial^* E \cap (D_1 \times (t,1))$$

Letting now $t \to (-1)^+$, since (9) holds we obtain:

$$\int_{D_1} \varphi = \int_M \varphi(px)\, (e_n \cdot \nu_E(x))\, d\mathcal{H}^{n-1}. \quad \blacksquare$$