Lecture 29

The Lipschitz approximation theorem.

The goal of this theorem is to show that if E is a minimizer in $C(x_0, 9r, \nu)$ and $e(E, x_0, 9r, \nu) << 1$, then $\partial E \cap C(x_0, r, \nu)$ is covered by the graph of a Lipschitz function up to an error controlled by $e(E, x_0, 9r, \nu)$.

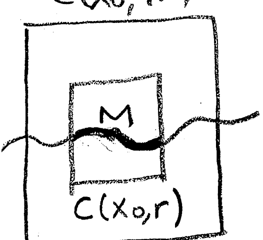
We prove the Theorem for $\nu = e_n$

<u>Thm (Lipschitz approximation)</u>: $\exists\, C_1(n), \varepsilon_1(n), \delta_0(n)$ s.t. if

- E minimizes perimeter in $C(x_0, 9r)$, $x_0 \in \partial E$
- $e(E, x_0, 9r, e_n) \le \varepsilon_1(n)$
- if we set $M = C(x_0 r) \cap \partial E$, $M_0 = \{ y \in M : \sup\limits_{0<s<8r} e(E, x_0, s, e_n) \le \delta_0(n) \}$

<u>Then</u>: $\exists\, u : \mathbb{R}^{n-1} \to \mathbb{R}$ Lipschitz s.t.:

C(x_0, 9r)



$M_0 \subset M \cap \Gamma$

- $\sup\limits_{\mathbb{R}^{n-1}} \dfrac{|u|}{r} \le C_1(n)\, e(E, x_0, 9r, e_n)^{\frac{1}{2(n-1)}}$, $\mathrm{Lip}(u) \le 1$
- $M_0 \subset M \cap \Gamma$, $\Gamma = x_0 + \{ (z, u(z)) : z \in D_r \}$
- $\dfrac{\mathcal{H}^{n-1}(M \Delta \Gamma)}{r^{n-1}} \le C_1(n)\, e(E, x_0, 9r, e_n)$
- $\dfrac{1}{r^{n-1}} \displaystyle\int_{D_r} |\nabla' u|^2 \le C_1(n)\, e(E, x_0, 9r, e_n)$
- $\dfrac{1}{r^{n-1}} \left| \displaystyle\int_{D_r} \nabla' u \cdot \nabla' \varphi \right| \le C_1(n) \sup\limits_{D_r} |\nabla' \varphi|\, e(E, x_0, 9r, e_n)$, $\forall \varphi \in C^1_c(D_r)$

<u>Remark</u>: The last two inequalities in the theorem say that u is "almost harmonic". This is because, roughly speaking, since E minimizes perimeter, then u, up to a rescaled error of size $\mathcal{H}^{n-1}(M \Delta \Gamma)$, is a local minimizer of the area functional. However, in the small gradient regime, the area functional and the Dirichlet integral are close; that is:

$$\int_{D_r} \sqrt{1+|\nabla' u|^2} \approx \mathcal{H}^{n-1}(D_r) + \frac{1}{2}\int_{D_r} |\nabla' u|^2$$

For this reason, the deviation of u from being harmonic turns out to be controlled in t

<u>Proof</u>:

<u>Step one</u>: Up to replacing E with $E_{x_0,r}$ and u with:

$$u_r(z) = \frac{1}{r} u(rz), \quad z \in \mathbb{R}^{n-1}$$

we can reduce the problem to proving the following:

$\exists C_1(n),\ \varepsilon_1(n),\ \delta_0(n)$ s.t. if

- E minimizes perimeter in $C(0,9) = C_9$
  $0 \in \partial E$
- $e(E,0,9,e_n) \leq \varepsilon_1(n)$
- If we set $M = C_1 \cap \partial E$, $M_0 = \{y \in M : \sup\limits_{0<s<8} e(E,0,s,e_n) \leq \delta_0(n)\}$

Then: $\exists u : \mathbb{R}^{n-1} \to \mathbb{R}$ Lipschitz s.t

- $\sup\limits_{\mathbb{R}^{n-1}} |u| \leq C_1(n) e(E,0,9,e_n)^{\frac{1}{2(n-1)}}$
- $M_0 \subset M \cap \Gamma$, $\quad \Gamma = \{(z,u(z)) : z \in D_1\}$
- $\mathcal{H}^{n-1}(M \Delta \Gamma) \leq C_1(n) e(E,0,9,e_n)$
- $\int_{D_1} |\nabla' u|^2 \leq C_1(n) e(E,0,9,e_n)$
- $\left| \int_{D_1} \nabla' u \cdot \nabla' \varphi \right| \leq C_1(n) \sup\limits_{D_1} |\nabla' \varphi| \, e(E,0,9,e_n), \forall \varphi \in C_c^1(D_1)$

Let $\varepsilon_0(n)$ and $C_0(n)$ denote the constants determined in the <u>Height bound Theorem</u>, proved in previous lecture. If we inmediately impose that $\varepsilon_1(n) \leq \varepsilon_0(n)$, then by the height bound theorem:

$$\boxed{\sup\{|q x| : x \in C_2 \cap \partial E\} \leq C_0(n) e(E,0,9,e_n)^{\frac{1}{2(n-1)}} \qquad (1)}$$

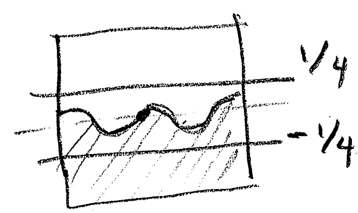
Recall also that from the Lemmas "Small-excess position" and "Excess measure" proved in Lecture 27, we have

$$\boxed{0 \le \mathcal{H}^{n-1}(M \cap p^{-1}(G)) - \mathcal{H}^{n-1}(G) \le e(E,0,1,e_n) \le 9^{n-1} e(E,0,9,e_n)}$$

for every Borel set $G \subset D$, (2)

Also, since, by construction, $\varepsilon_0(n) \le \omega(n, \frac{1}{4})$, Lemma "small-excess position" implies that:

$$\{x \in C_2 : qx < -\tfrac{1}{4}\} \subset C_2 \cap E \subset \{x \in C_2 : qx < \tfrac{1}{4}\}$$



<u>Step two</u>: We show that M_0 is contained in the graph of a Lipschitz function u.

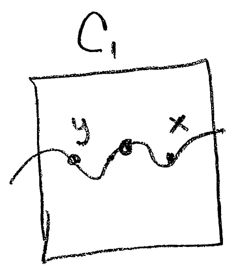
We fix $y \in M_0$, $x \in M$ and consider the blow-up of E at scale $\|y-x\|_\infty$ centered at y; that is,

$$F = E_{y, \|y-x\|_\infty} = \frac{E-y}{\|y-x\|_\infty}$$

where $\|\cdot\|_\infty$ is defined as:

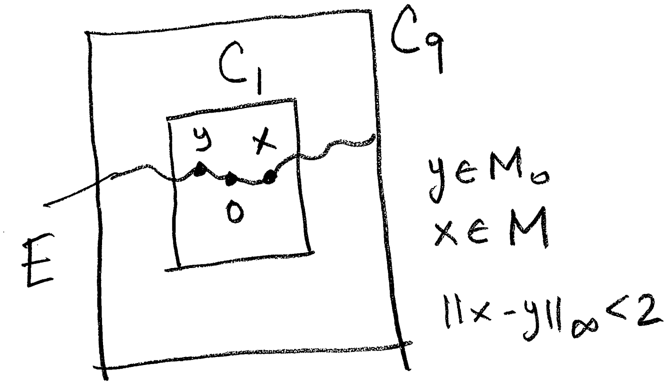
$\|z\|_\infty = \max\{|pz|, |qz|\}$, $z \in \mathbb{R}^n$, so that:

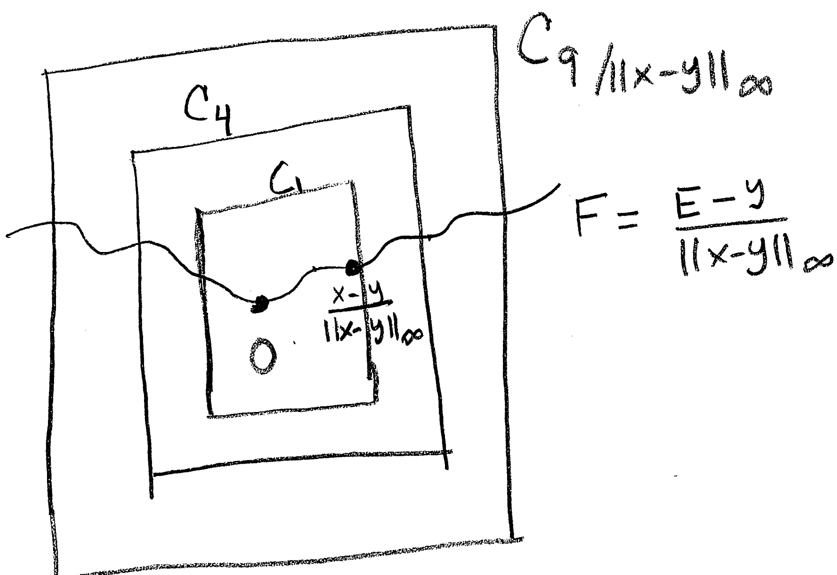
$$C(y,s) = \{z : \mathbb{R}^n : \|z-y\|_\infty < s\}.$$



$$\|y-x\|_\infty \le 2$$

By Remark 1 in Lecture 28, we have F is a perimeter minimizer in $C_{9/\|x-y\|_\infty}$.





Note that:

$4\|x-y\|_\infty < 8$

$0 \in \partial F$

Recall, from remark 2 in Lecture 28 that the excess does not change under blow-ups; that is,

$$e(F, 0, C_4, e_n) = e(E, y, 4\|x-y\|_\infty, e_n)$$

$$\leq \delta_0(n) \quad ; \quad \text{since } 4\|x-y\|_\infty < \infty; \text{ and } y \in M_0$$

then, provided we impose:

$$\boxed{\delta_0(n) \leq \varepsilon_0(n)}$$

we can now apply the height bound theorem to obtain (recall that $\varepsilon_0(n)$ is the constant found in the height bound theorem):

$$\sup\{ |qw| : w \in C_1 \cap \partial F \} \le C_0(n)\, \delta_0(n)^{\frac{1}{2(n-1)}}.$$

We can now test this condition on $w = \frac{x-y}{\|x-y\|_\infty}$. Notice that $\|w\|_\infty = 1$ and $\frac{x-y}{\|x-y\|_\infty} \in \partial F$.

$$\therefore \left| q\left(\frac{x-y}{\|x-y\|_\infty} \right) \right| \le C_0(n)\, \delta_0(n)^{\frac{1}{2(n-1)}}$$

$$\Rightarrow |qx - qy| \le C_0(n)\, \delta_0(n)^{\frac{1}{2(n-1)}} \, \|x-y\|_\infty .$$

We now choose $\delta_0(n)$ even smaller to ensure that:

$$L(n) := C_0(n)\, \delta_0(n)^{\frac{1}{2(n-1)}} < 1$$

Thus:

$$|qx - qy| < \underbrace{\|x-y\|_\infty}_{= \max\{ |qx-qy|,\ |px-py| \}}$$

$\therefore \|x-y\|_\infty = |px - py|$, and hence:

$$\boxed{|qx - qy| \le L(n)\, |px - py|, \ \forall x, y \in M_0 \qquad (4)}$$

Notice that, by (4):

$$px = py \implies qx = qy,$$

and then we can define a function:

$$u : p(M_0) \to \mathbb{R}$$
$$u(px) = qx, \quad x \in M_0$$

and, with this definition:

$$\boxed{|u(py) - u(px)| \le L(n)\,|py - px|, \quad \forall x, y \in M_0}$$

Since $M_0 \subset C_2 \cap \partial E$ from (1), we also have:

$$(5) \quad |u(px)| \le C_0(n)\, e(E, 0, 9, e_n)^{\frac{1}{2(n-1)}}, \quad \forall x \in M_0$$

We can now extend u from $p(M_0)$ to a Lipschitz function:

$$\boxed{u : \mathbb{R}^{n-1} \to \mathbb{R}, \quad \mathrm{Lip}(u) \le L(n) < 1, \quad M_0 \subset \Gamma = \{(z, u(z)) : z \in D_1\}}$$

Moreover, from (5) and, up to truncating u, we have:

$$\boxed{\sup_{\mathbb{R}^{n-1}} |u| \le C_0(n)\, e(E, 0, 9, e_n)^{1/2(n-1)}}$$

We have thus proved the first two statements of the Theorem.

We now proceed to estimate:

$$\mathcal{H}^{n-1}(M \Delta \Gamma),$$

which is the part of $M = \partial E \cap C$, that is not covered with the graph of the Lipschitz function u.

By definition of M_0, for every $y \in M \setminus M_0$, $\exists s_y \in (0, 8)$ such that:

$$\boxed{\delta_0(n)\, s_y^{n-1} < \int_{C(y,s_y)\cap\partial E} \frac{|\nu_E - e_n|^2}{2}\, d\mathcal{H}^{n-1},} \quad (5)$$

Now, we consider the collection of balls $\{B(y, \sqrt{2} s_y)\}_{y \in M \setminus M_0}$. By Besicovitch Covering theorem $\exists\, \beta(n)$ and disjoint subfamilies $\mathcal{F}_1, \dots, \mathcal{F}_{\beta(n)}$ s.t:

$$M \setminus M_0 \subset \bigcup_{i=1}^{\beta(n)} \bigcup_{B \in \mathcal{F}_i} B$$

$$\mathcal{H}^{n-1}(M \setminus M_0) \leq \sum_{i=1}^{\beta(n)} \mathcal{H}^{n-1}\Big((M \setminus M_0) \cap \Big(\bigcup_{B \in \mathcal{F}_i} B\Big)\Big)$$

Choose $i \in \{1, 2, \dots, \beta(n)\}$ such that:

$$\mathcal{H}^{n-1}(M \setminus M_0) \leq \beta(n)\, \mathcal{H}^{n-1}\Big(M \setminus M_0 \cap \Big(\bigcup_{B \in \mathcal{F}_i} B\Big)\Big)$$

$$= \beta(n) \sum_{K=1}^{\infty} \mathcal{H}^{n-1}(M \setminus M_0 \cap B_K)$$

where $\mathcal{F}_i = \{B_K(y_K, \sqrt{2} s_K)\}_{K=1}^{\infty}$ are disjoint.

Recall that:

$$E \text{ minimizer} \Rightarrow \omega_{n-1} \le \frac{P(E; B(x,r))}{r^{n-1}} \le n\omega_n,$$

$\forall x \in \partial E$

Therefore:

$$\boxed{\begin{aligned} \mathcal{H}^{n-1}(M \setminus M_0) &\le \beta(n) \sum_{k=1}^{\infty} n\omega_n \sqrt{2}^{\,n-1} s_k^{n-1} \\ &= c(n) \sum_{k=1}^{\infty} s_k^{n-1}. \end{aligned}}$$

Since the cylinders $\{C(y_k, s_k)\}$ are mutually disjoint and contained in C_q:

$$\mathcal{H}^{n-1}(M \setminus M_0) \le c(n) \sum_{k=1}^{\infty} \frac{1}{\delta_0(n)} \int_{C(y_k, s_k) \cap \partial E} \frac{|\nu_E - e_n|^2}{2} \, d\mathcal{H}^{n-1}; \text{ by } (5)$$

$$= C(n) \frac{1}{\delta_0(n)} \sum_{k=1}^{\infty} \int_{C(y_k, s_k) \cap \partial E} \frac{|\nu_E - e_n|^2}{2} \, d\mathcal{H}^{n-1}$$

$$\le C(n) \frac{1}{\delta_0(n)} \int_{\partial E \cap C_q} \frac{|\nu_E - e_n|^2}{2} \, d\mathcal{H}^{n-1}$$

$$= \underbrace{\frac{C(n)}{\delta_0(n)} q^{n-1}}_{C(n)} \cdot \frac{1}{q^{n-1}} \int_{\partial E \cap C_q} \frac{|\nu_E - e_n|^2}{2} \, d\mathcal{H}^{n-1}$$

$$= C(n)\, e(E, 0, q, e_n).$$

We have proved:

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$$\boxed{\mathcal{H}^{n-1}(M \setminus M_0) \leq c(n)\, e(E,0,9,e_n)}$$

Since $M \setminus \Gamma \subset M \setminus M_0$ we have:

$$\boxed{\mathcal{H}^{n-1}(M \setminus \Gamma) \leq c(n)\, e(E,0,9,e_n)}$$

Since $M \Delta \Gamma = (M \setminus \Gamma) \cup (\Gamma \setminus M)$, we are left to bound $\mathcal{H}^{n-1}(\Gamma \setminus M)$,

We have:

$$\mathcal{H}^{n-1}(\Gamma \setminus M) = \int_{p(\Gamma \setminus M)} \sqrt{1 + |\nabla' u|^2}\, dz$$

$$\leq \sqrt{1 + \mathrm{Lip}(u)^2} \int_{p(\Gamma \setminus M)} dz \quad ; \quad \text{since } |\nabla' u| \leq \mathrm{Lip}\, u$$

$$\leq \sqrt{2}\, \mathcal{H}^{n-1}(P(\Gamma \setminus M)) \, ; \quad \mathrm{Lip}(u) \leq 1, \ dz = \mathcal{H}^{n-1}$$

Since $M \cap p^{-1}(p(\Gamma \setminus M)) \subset M \setminus \Gamma$, we have:

$$\begin{aligned}
\mathcal{H}^{n-1}(\Gamma \setminus M) &\leq \sqrt{2}\, \mathcal{H}^{n-1}(p(\Gamma \setminus M)) \\
&\leq \sqrt{2}\, \mathcal{H}^{n-1}(M \cap p^{-1}(p(\Gamma \setminus M))) \\
&\leq \sqrt{2}\, \mathcal{H}^{n-1}(M \setminus \Gamma) \\
&\leq C(n)\, e(E,0,9,e_n).
\end{aligned}$$

We conclude that $\boxed{\mathcal{H}^{n-1}(M \Delta \Gamma) \leq c(n)\, e(E,0,9,e_n)}$