Lecture 3

Hausdorff measures

Measure-theoretic notion of dimension.

$$\dim(E) = \inf\{s \in [0,\infty) : \mathcal{H}^s(E) = 0\}$$

We have:

$$H^s(\lambda A) = \lambda^s H^s(A), \quad \forall \lambda > 0, \ \forall A$$

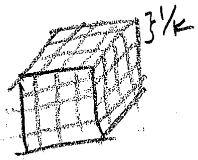
$$\mathcal{H}^s(LA) = \mathcal{H}^s(A), \quad \forall L : \mathbb{R}^n \to \mathbb{R}^n \text{ linear isometry}$$

<u>Proposition</u>: If $s > n$, then $\mathcal{H}^s \equiv 0$

Let $Q = (0,1)^n$. Since $\lambda^s \mathcal{H}^s(Q) = \mathcal{H}^s(\lambda Q) \to \mathcal{H}^s(\mathbb{R}^n)$ as $\lambda \to \infty$, it suffices to prove $\mathcal{H}^s(Q) = 0$.

Consider a partition of Q by <u>K^n cubes</u> of diameter $\frac{\sqrt{n}}{K}$

<u>Proof</u>:

Note that: 

$$\sqrt{\underbrace{\left(\tfrac{1}{K}\right)^2 + \cdots + \left(\tfrac{1}{K}\right)^2}_{n \text{ times}}} = \sqrt{\frac{n}{K^2}} = \frac{\sqrt{n}}{K}$$

$$\mathcal{H}^s_{\frac{\sqrt{n}}{K}}(Q) \le \omega_s K^n \left(\frac{\sqrt{n}}{2K}\right)^s = \omega_s \frac{n^{s/2}}{2^s} K^{n-s} \to 0$$

as $K \to \infty$, because $n - s < 0$. ∎

<u>Prop</u>: If $E \subset \mathbb{R}^n$, then $\dim(E) \in [0,n]$, and $\mathcal{H}^s(E) = \infty$, for every $s < \dim(E)$.

Proof: Since $\mathcal{H}^s(E) = 0$ $\forall s > 0$, then $0 \leq \dim(E) \leq n$.

Now,
<u>if $\mathcal{H}^s(E) < \infty$ for some $s \in [0,n) \Rightarrow \mathcal{H}^t(E) = 0$, $\forall t > s$,</u>

Indeed, fix $\delta > 0$ and let $\{F_i\}$ so that $\sum \omega_s \left(\frac{\operatorname{diam} F_i}{2}\right)^s \leq \mathcal{H}^s_\delta(E) + 1 \leq \mathcal{H}^s(E) + 1$, $(\operatorname{diam} F_i \leq \delta)$

$$\Rightarrow \mathcal{H}^t_\delta(E) \leq \omega_t \sum_{i=1}^{\infty} \left(\frac{\operatorname{diam} F_i}{2}\right)^t$$

$$\leq \left(\frac{\delta}{2}\right)^{t-s} \frac{\omega_t}{\omega_s} \omega_s \sum_{i=1}^{\infty} \left(\frac{\operatorname{diam} F_i}{2}\right)^s$$

$$\leq C(t,s)\, \delta^{t-s} (\mathcal{H}^s(E) + 1).$$

Letting $\delta \to 0$, since $t - s > 0$ we get:

$$\mathcal{H}^t(E) = 0. \quad \blacksquare$$

<u>Corollary</u>: If $E \subset \mathbb{R}^n$ and $0 < \mathcal{H}^s(E) < \infty$, then $s = \dim(E)$. The converse is not necessarily true: it may happen $\mathcal{H}^s(E) \in \{0, \infty\}$, $s = \dim(E)$

<u>Note</u>: If $\mathcal{H}^t(E) > 0$ then $\mathcal{H}^s(E) = \infty$, $\forall s < t$.

<u>Proposition</u>: $\mathcal{H}^0$ is the counting measure

Proof: $x \in \mathbb{R}^n$, $\delta > 0$ $\Rightarrow \mathcal{H}^0_\delta(\{x\}) = \omega_0 = 1. \Rightarrow \mathcal{H}^0(\{x\}) = 1.$

Thus; since $\mathcal{H}^0$ is Borel (σ-additivity holds):

$$\mathcal{H}^0(E) = \sum_{x \in E} \mathcal{H}^0(\{x\})$$
$$= \# E, \quad \text{whenever } E \text{ is finite or countable.}$$

If E is infinite, then $\exists\, F \subset E$, F countable, and hence

$$\mathcal{H}^0(E) \geq \mathcal{H}^0(F) = \infty \quad (\text{by monotonicity}). \blacksquare$$

<u>Proposition</u>: If $E \subset \mathbb{R}^n$ and $\mathcal{H}^s_\infty(E) = 0$, then $\mathcal{H}^s(E) = 0$

Proof: $s=0$ trivial. Let $s>0$, if $\mathcal{H}^s_\infty(E)=0$ then $\forall\, \varepsilon > 0$ $\exists\, \{F_i\}$, $E \subset \bigcup F_i$, $\operatorname{diam} F_i < \delta$ such that:

$$\omega_s \sum_{i=1}^{\infty} \left(\frac{\operatorname{diam} F_i}{2}\right)^s \leq \varepsilon,$$

thus,

$$\operatorname{diam}(F_i) \leq 2\left(\frac{\varepsilon}{\omega_s}\right)^{1/s} = \delta(\varepsilon), \quad \forall i.$$

Hence $\mathcal{H}^s_{\delta(\varepsilon)}(E) \leq \varepsilon$ with $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. $\blacksquare$

<u>Hausdorff measures and Lipschitz functions</u>:

$$\operatorname{Lip}(f; E) := \inf\{L : |f(x) - f(y)| \leq L|x-y| \ \forall x, y \in E\}$$

$$\operatorname{Lip}(f; \mathbb{R}^n) = \operatorname{Lip}(f).$$

<u>Proposition</u>: If $f: \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz function, then:

$$\mathcal{H}^s(f(E)) \leq \mathrm{Lip}(f)^s \mathcal{H}^s(E),$$

for every $s > 0$, $E \subset \mathbb{R}^n$. In particular, $\dim(f(E)) \leq \dim(E)$.

Proof: Let $\{F_i\}$, $E \subset \bigcup_{i=1}^{\infty} F_i$, $\operatorname{diam} F_i < \delta$.

$$\Rightarrow \quad \operatorname{diam} f(F_i) \leq \mathrm{Lip}(f) \operatorname{diam}(F_i)$$
$$\leq \mathrm{Lip}(f)\,\delta$$

$$\Rightarrow \mathcal{H}^s_{\mathrm{Lip}(f)\delta}(f(E)) \leq \omega_s \sum_{i=1}^{\infty} \left(\frac{\operatorname{diam} f(F_i)}{2}\right)^s$$

$$\leq \mathrm{Lip}(f)^s \omega_s \sum_{i=1}^{\infty} \left(\frac{\operatorname{diam} F_i}{2}\right)^s,$$

Since $\{F_i\}$ is arbitrary, we get:

$$\mathcal{H}^s_{\mathrm{Lip}(f)\delta}(f(E)) \leq \mathrm{Lip}(f)^s \mathcal{H}^s_\delta(E)$$

Let $\delta \to 0^+ \quad \Rightarrow \quad \mathcal{H}^s(f(E)) \leq \mathrm{Lip}(f)^s \mathcal{H}^s(E)$

<u>Ex</u>: Let $\pi: \mathbb{R}^n \to \mathbb{R}^m$, projection map.

π is Lipschitz, $\mathrm{Lip}(\pi) \leq 1$,

so by previous result:

$$\mathcal{H}^s(\pi(E)) \leq \mathcal{H}^s(E)$$

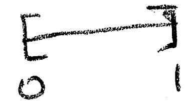
$\mathcal{H}^1$ and the classical notion of length:

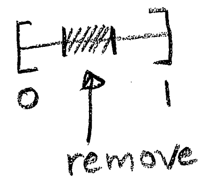
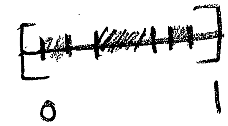
A set $\Gamma \subset \mathbb{R}^n$ is a curve if $\exists a > 0$ and a continuous, injective function $\gamma : [0,a] \to \mathbb{R}^n$ such that $\Gamma = \gamma([0,a])$. γ is called a parametrization of Γ. The length of Γ is defined as:

$$\text{length}(\Gamma) = \sup\left\{ \sum_{i=1}^{N} |\gamma(t_i) - \gamma(t_{i-1})| : 0 = t_0 < t_{i-1} < t_i < t_N = a \right\}$$

<u>Theorem</u>: $\text{length}(\Gamma) = \mathcal{H}^1(\Gamma)$. If Γ is C^1 then $\mathcal{H}^1(\Gamma) = \int_0^1 |\gamma'(t)|\,dt$.

<u>Remark</u>: Consider the standard Cantor set:

$C = \bigcap_{K=1}^{\infty} C_K$, $C_1 =$ 

$C_2 =$ , $C_3 =$ 

It can be proven that $\dim(C) = \frac{\log 2}{\log 3} = s$, and $\mathcal{H}^s(C) = 1$. Moreover, for every $s \in [0,n]$, $\exists K \subset \mathbb{R}^n$ such that $\dim(K) = s$

We now proceed to show that:

Theorem 1: $\mathcal{H}^n = \mathcal{L}^n$

We will need the following:

Theorem 2 (Isodiametric inequality): Among all sets of fixed diameter, balls have maximum volume. That is,

$$|E| \leq \omega_n \left(\frac{\operatorname{diam}(E)}{2} \right)^n, \quad \forall E \subset \mathbb{R}^n.$$

In order to prove theorem 1 we need two inequalities:

$$|E| \leq \mathcal{H}^n(E) \qquad (1)$$

$$\mathcal{H}^n(E) \leq |E| \qquad (2)$$

(1) follows from the isodiametric inequality. Indeed, given $\delta \in (0, \infty]$, let $\{F_i\}$, $E \subset \bigcup F_i$, $\operatorname{diam}(F_i) \leq \delta$. By monotonicity:

$$|E| \leq \left| \bigcup_{i=1}^{\infty} F_i \right| \leq \sum_{i=1}^{\infty} |F_i|$$

$$\leq \omega_n \sum_{i=1}^{\infty} \left(\frac{\operatorname{diam}(F_i)}{2} \right)^n; \quad \text{by Isodiametric inequality}$$

By the arbitrariness of $\{F_i\}$ we have:

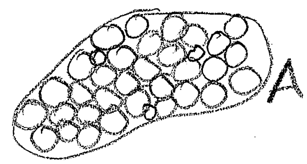
$$|E| \leq \mathcal{H}^n_\delta(E)$$

Letting $\delta \to 0$ we get $|E| \leq \mathcal{H}^n(E)$.

To prove (2) we will use:

Vitali's property of Lebesgue measure: If $A \subset \mathbb{R}^n$ is open, $\delta > 0$. $\Rightarrow \exists \{\bar{B}_i\}$ disjoint closed balls $\bar{B}_i \subset A$, $\forall i$, $\operatorname{diam}(\bar{B}_i) < \delta$ such that:

$$\left| A \setminus \bigcup_{i=1}^{\infty} \bar{B}_i \right| = 0$$



$F = \bigcup_{i=1}^{\infty} \bar{B}_i$ $\qquad |A| \leq |E| + \varepsilon$

$|A| = |F|$

We want $|E| \geq \mathcal{H}^n(E)$; assume $|E| = \infty$.
Given $\varepsilon, \delta > 0$, let A open $E \subset A$, $|A| \leq |E| + \varepsilon$.
By Vitali's property, we cover A with a countable collection of closed balls $\{\bar{B}_i\}$, $\operatorname{diam} \bar{B}_i < \delta$, $|A \setminus \bigcup_{i=1}^{\infty} \bar{B}_i| = 0$. Then, if $F := \bigcup_{i=1}^{\infty} \bar{B}_i$ we have.

$$\mathcal{H}^n_\delta(F) \leq \sum_{i=1}^{\infty} \omega_n \left(\frac{\operatorname{diam} \bar{B}_i}{2} \right)^n$$

$$= \sum_{i=1}^{\infty} |\bar{B}_i| = \left| \bigcup_{i=1}^{\infty} \bar{B}_i \right| = |A| \leq |E| + \varepsilon$$

(Note: Easy to check: $\mathcal{H}^n_\infty(E) \leq \omega_n \left(\frac{\sqrt{n}}{2} \right)^n |E|$
Indeed, Let $\{F_i\}$ be a covering of CUBES of E in the definition of $|E|$, then $\operatorname{diam}(F_i) = \sqrt{n}\, r(F_i)$
$\Rightarrow \mathcal{H}^n_\infty(E) \leq \omega_n \sum_{i=1}^{\infty} \left(\frac{\operatorname{diam} F_i}{2} \right)^n = \omega_n \left(\frac{\sqrt{n}}{2} \right)^n \sum_{i=1}^{\infty} r(F_i)^n$, and $\{F_i\}$ arbitrary.)

Going back, we have;

$$\boxed{\mathcal{H}^n_\delta(F) \le |E| + \varepsilon}$$

Also, by previous note:

$$\mathcal{H}^n_\infty(A \setminus F) \le \omega_n\left(\frac{\sqrt{n}}{2}\right)^n |A \setminus F| = 0$$

$$\Rightarrow \mathcal{H}^n_\infty(A \setminus F) = 0 \quad \Rightarrow \mathcal{H}^n(A \setminus F) = 0$$

$$\Rightarrow \boxed{\mathcal{H}^n_\delta(A \setminus F) = 0}$$

(Note: Recall that we proved earlier that:

$E \subset \mathbb{R}^n$, $\mathcal{H}^s_\infty(E) = 0 \Rightarrow \mathcal{H}^s(E) = 0$

and $\mathcal{H}^s(E) = \sup_{\delta \in (0,\infty]} \mathcal{H}^s_\delta(E) = \lim_{\delta \to 0^+} \mathcal{H}^s_\delta(E)$)

Thus,

$$\begin{aligned} \mathcal{H}^n_\delta(E) &\le \mathcal{H}^n_\delta(E \cap F) + \mathcal{H}^n_\delta(E \setminus F) \\ &\le \mathcal{H}^n_\delta(F) + \mathcal{H}^n_\delta(A \setminus F); \quad E \cap F \subset F,\ E \setminus F \subset A \setminus F \\ &\le |E| + \varepsilon + 0 \end{aligned}$$

Letting $\varepsilon, \delta \to 0^+$ we conclude;

$$\mathcal{H}^n(E) \le |E|.$$

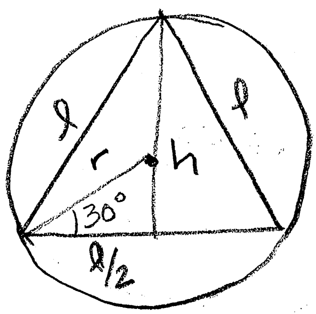
(3.9)

<u>Proof of the Isodiametric inequality</u>:

$$|E| \le \omega_n \left(\frac{\mathrm{diam}(E)}{2}\right)^n, \quad E \subset \mathbb{R}^n, \quad |\omega_n| = |B_1|$$

Clearly true if E is contained in a ball of the same diameter

<u>Ex</u>: Consider an equilater triangle. The triangle is not contained in any ball of the same diameter



$\sin 60^\circ = \frac{h}{l}$

$h = l \sin 60^\circ$

$h = \frac{l\sqrt{3}}{2}$

$\cos 30^\circ = \frac{l/2}{r}$

$r = \frac{l/2}{\cos 30^\circ} = \frac{l}{\sqrt{3}}$

$\mathrm{diam}(\text{Circle}) = \frac{2l}{\sqrt{3}}$

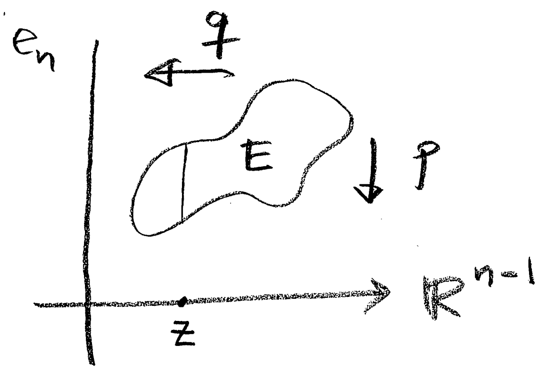
$\mathrm{diam}(\text{triangle}) < \mathrm{diam}(\text{Circle})$.

"The ball has maximum volume among all sets with a fixed diameter".

<u>Steiner symmetrization with respect to e_n</u>.

Assume E is closed

Consider the projections $p: \mathbb{R}^n \to \mathbb{R}^{n-1}$, $p(z,t) = z$, $q: \mathbb{R}^n \to \mathbb{R}$, $q(z,t) = t$, $x = (z,t)$, $z \in \mathbb{R}^{n-1}$, $t \in \mathbb{R}$.

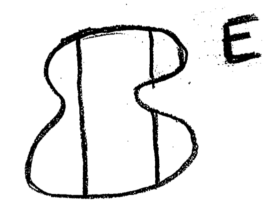


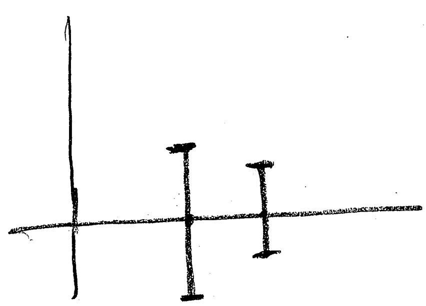
For $z \in \mathbb{R}^{n-1}$, define:

$$E_z = \{t \in \mathbb{R} : (z, t) \in E\}$$

Define:

$$E^s = \left\{x \in \mathbb{R}^n : |qx| \le \frac{\mathcal{L}^1(E_{px})}{2}\right\}$$



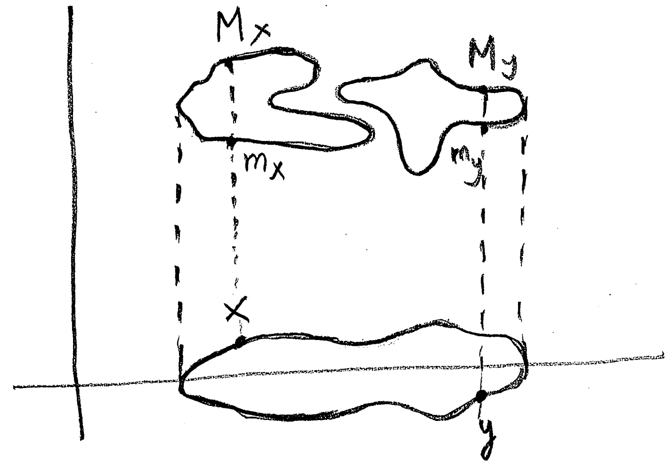


By Fubini, $z \mapsto \mathcal{L}^1(E_z)$ is measurable,

and

$$|E| = \int_{\mathbb{R}^{n-1}} \mathcal{L}^1(E_z)\, dz = |E^s|$$

Claim: $\operatorname{diam} E^s \le \operatorname{diam} E$



$$|qx - qy| \le \max\{|qM_x - qm_y|, |qm_x - qM_y|\}$$

$$\left.\begin{array}{l} pM_x = pm_x = px \\ pM_y = pm_y = py \end{array}\right\}$$ Note that: $|x|^2 = |px|^2 + |qx|^2$.

$$\therefore |x - y| \le \max\{|M_x - m_y|, |M_y - m_x|\}$$

WLOG E is closed (otherwise, replace E by $\bar{E}$):

Apply Steiner symmetrization to E with respect to $e_1, \dots, e_n$ and let F be the resulting set.

F is symmetric w.r.t. $e_1, \dots, e_n$, $|E| = |F|$, $\operatorname{diam}(F) \le \operatorname{diam}(E)$

$F = -F$ since F is symmetric w.r.t. $e_1, \dots, e_n$.

$$F = -F \Rightarrow F \subset B_{\frac{\operatorname{diam} F}{2}}(0) \Rightarrow |F| \le \left|B_{\frac{\operatorname{diam} F}{2}}(0)\right| \le |B_1(0)| \left(\frac{\operatorname{diam} F}{2}\right)^n. \blacksquare$$