Lecture 30

In this Lecture we continue with the proof of the Lipschitz approximation theorem. We now proceed to prove the almost harmonicity of the Lipschitz function u.

We recall from Lecture 9, Page 9.7, that if M_1 and M_2 are two locally $\mathcal{H}^k$-rectifiable sets in $\mathbb{R}^n$, then for $\mathcal{H}^k$- a.e. $x \in M_1 \cap M_2$,

$$T_x M_1 = T_x M_2$$

Moreover, if $u: \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lispchitz function and $f: \mathbb{R}^{n-1} \to \mathbb{R}^n$ is given by $f(z) = (z, u(z))$, $z \in \mathbb{R}^{n-1}$, then $\Gamma = f(\mathbb{R}^{n-1})$ is locally $\mathcal{H}^{n-1}$-rectifiable and, for a.e. $z \in \mathbb{R}^{n-1}$,

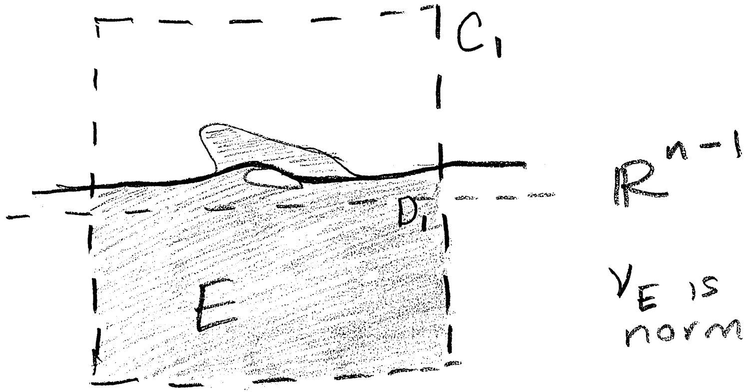
$$T_{f(z)} \Gamma = \nu(z)^{\perp}, \quad \nu(z) = (-\nabla' u(z), 1)$$

Also, recall that if E is a set of locally finite perimeter then:

$$T_x(\partial^* E) = \nu_E(x)^{\perp}, \quad \forall x \in \partial^* E$$

Therefore, going back to our setting, we have $M = \partial E \cap C$, and we have covered the good set M_0, $M_0 \subset M$, with the graph of a Lipschitz function $u : \mathbb{R}^{n-1} \to \mathbb{R}$, $\mathrm{Lip}(u) < 1$. Thus, from the previous, discussion, with $M_1 = \partial E$, $M_2 = \Gamma$, we deduce that: (30.2)

$$\nu_E(x) = \lambda(x) \frac{(-\nabla' u(px), 1)}{\sqrt{1 + |\nabla' u(px)|^2}}, \quad \lambda(x) = \pm 1$$



ν_E is the outer normal to the set.

We can now estimate:

$$\int_{p(M\cap\Gamma)} |\nabla' u|^2 \leq \sqrt{2} \int_{p(M\cap\Gamma)} \frac{|\nabla' u(z)|^2}{\sqrt{1 + |\nabla' u(z)|^2}} \, dz \,;$$

because $\mathrm{Lip}(u) \leq 1$ and hence $|\nabla' u(z)| \leq 1 \Rightarrow \sqrt{1 + |\nabla' u(z)|^2} \leq \sqrt{2}$

$$= \sqrt{2} \int_{M\cap\Gamma} \frac{|\nabla' u(px)|^2}{\sqrt{1 + |\nabla' u(px)|^2}} (\nu_E \cdot e_n) \, d\mathcal{H}^{n-1} \,;$$

by the excess measure theorem we have:

$$\int_{D_1} \varphi = \int_M \varphi(px) (\nu_E(x) \cdot e_n) \, d\mathcal{H}^{n-1}(x)$$

$\forall \varphi \in C_c(D_1)$

$$\leq \sqrt{2} \int_{M\cap\Gamma} \frac{|\nabla' u(px)|^2}{1 + |\nabla' u(px)|^2} \, d\mathcal{H}^{n-1} \,;$$

since

$$|\nu_E(x) \cdot e_n| = \frac{1}{\sqrt{1 + |\nabla' u(px)|^2}}$$

30.3

Hence, we have:

$$\boxed{\int_{p(M\cap\Gamma)} |\nabla' u|^2 dz \leq \sqrt{2} \int_{M\cap\Gamma} \frac{|\nabla' u(px)|^2}{1+|\nabla' u(px)|^2} d\mathcal{H}^{n-1} \qquad (1)}$$

Also, notice that:

$$\frac{|p\nu_E|^2}{2} = \frac{1-(\nu_E\cdot e_n)^2}{2}$$

$$= \frac{(1+\nu_E\cdot e_n)(1-\nu_E\cdot e_n)}{2}$$

$$\leq 1-\nu_E\cdot e_n \quad ; \quad \text{since } \frac{1+\nu_E\cdot e_n}{2} \leq \frac{1+1}{2} = 1$$

$$= \frac{|\nu_E - e_n|^2}{2}$$

Moreover:

$$\frac{|\nabla' u(px)|^2}{1+|\nabla' u(px)|^2} = |p\nu_E|^2 \; ; \quad \text{since } \nu_E(x) = \lambda(x)\frac{(-\nabla' u(px), 1)}{\sqrt{1+|\nabla' u(px)|^2}}$$

Hence, from (1):

$$\int_{p(M\cap\Gamma)} |\nabla' u|^2 dz \leq \sqrt{2} \int_{M\cap\Gamma} |p\nu_E|^2 d\mathcal{H}^{n-1}$$

$$= 2\sqrt{2} \int_{M\cap\Gamma} \frac{|p\nu_E|^2}{2} d\mathcal{H}^{n-1}$$

$$\leq 2\sqrt{2} \int_{M\cap\Gamma} \frac{|\nu_E - e_n|^2}{2}$$

$$= 2\sqrt{2}\, e(E,0,1,e_n) \leq 2\sqrt{2}\, 9^{n-1} e(E,0,9,e_n).$$

Also:

$$\int_{p(M\Delta\Gamma)} |\nabla' u|^2 dz \leq \mathcal{H}^{n-1}(p(M\Delta\Gamma)); \quad \text{since: } |\nabla' u| \leq 1$$

$$\leq \mathcal{H}^{n-1}(M\Delta\Gamma),$$

where we have used:

$$\begin{cases} f: \mathbb{R}^n \to \mathbb{R}^m \text{ Lipschitz function} \\ \text{Then:} \\ \quad \mathcal{H}^s(f(E)) \leq Lip(f)^s \mathcal{H}^s(E) \end{cases}$$

But we proved last class that:

$$\mathcal{H}^{n-1}(M\Delta\Gamma) \leq C(n)\, e(E, 0, 9, e_n)$$

Therefore:

$$\int_{p(M\Delta\Gamma)} |\nabla' u|^2 dz \leq c(n)\, e(E, 0, 9, e_n)$$

Thus, we have proved:

$$\int_{p(M\cap\Gamma)} |\nabla' u|^2 dz \leq 2\sqrt{2}\, 9^{n-1} e(E, 0, 9, e_n) \quad \underline{\text{and}} \quad \int_{p(M\Delta\Gamma)} |\nabla' u|^2 dz \leq c\, e(E, 0, 9, e_n)$$

We conclude that, since $D_1 = p(M)$:

$$\boxed{\int_{D_1} |\nabla' u|^2 dz \leq c_1(n)\, e(E, 0, 9, e_n)} \quad (*)$$

Now, since E minimizes perimeter in C_1, we have:

$$\boxed{\int_{\partial^* E} \operatorname{div}_E T = 0 \quad \forall\, T \in C_c^1(C_1; \mathbb{R}^n)} \quad (1)$$

Given $\varphi \in C_c^\infty(D_1)$, we define a vector field $T \in C_c^\infty(C_1; \mathbb{R}^n)$ by setting:

$$\begin{aligned} T(x) &= \alpha(qx)\,\varphi(px)\,e_n, \quad x \in \mathbb{R}^n \\ &= \alpha(x_n)\,\varphi(x_1, \dots, x_{n-1})\,e_n, \quad x \in \mathbb{R}^n \\ &= (0, \dots, 0, \alpha(x_n)\,\varphi(x_1, \dots, x_{n-1})) \end{aligned}$$

with $\alpha \in C_c^\infty((-1,1); [0,1])$, $\alpha(s) = 1$ for all $|s| < \frac{1}{4}$.

We recall that:

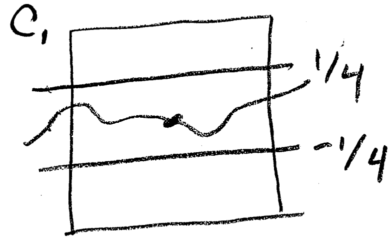
$$\operatorname{div}_E T = \operatorname{div} T - (\nabla T \nu_E) \cdot \nu_E$$

We compute:

$$\nabla T = \begin{pmatrix} 0, & \cdots & \cdots & 0 \\ 0, & \cdots & \cdots & 0 \\ & \vdots & & \vdots \\ \alpha(x_n)\varphi'(px) & \cdots & \alpha(x_n)\varphi'(px) & \dot{\alpha}(x_n)\varphi(px) \end{pmatrix}$$

We recall from previous class that, since we have choosen the bound $\varepsilon_0(n)$ in the height bound Theorem such that $\varepsilon_0(n) = \omega(n, \frac{1}{4})$, then by the Lemma "small-excess position" we have that:

$$M \equiv \partial E \cap C_1 \subset \{x \in C_1 : |qx| < \tfrac{1}{4}\}$$



then, when computing $\int_{\partial^* E} \operatorname{div}_E T \, d\mathcal{H}^{n-1}$, we only need to find $\operatorname{div}_E T$ for x, $|x| < \frac{1}{4}$.

Note that for every x, $|x| < \frac{1}{4}$, we have:

$$\nabla T(x) = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ & \vdots & & \\ \varphi'(px) & \cdots & \varphi'(px) & 0 \end{pmatrix}$$

since $\alpha(x_n) = 1$ and $\alpha'(x_n) = 0$ if $|x| < \frac{1}{4}$

$$\Rightarrow \quad \nabla T \, \nu_E = (0, \ldots, 0, \nabla' \varphi(px) \cdot \nu_E'),$$

where $\nu_E = (\nu_E', \nu_E \cdot e_n)$, and hence:

$$(\nabla T(x) \nu_E(x)) \cdot \nu_E(x) = (0, \ldots, 0, \nabla' \varphi(px) \cdot \nu_E') \cdot (\nu_E', \nu_E \cdot e_n)$$

$$= (\nabla' \varphi(px) \cdot \nu_E') \, \nu_E \cdot e_n$$

$$= \left[(\nabla' \varphi(px), 0) \cdot \nu_E \right] \nu_E \cdot e_n$$

Also, for x, $|x| < \frac{1}{4}$:

$$\operatorname{div} T(x) = 0,$$

$\Rightarrow$

$$\operatorname{div}_E T(x) = \operatorname{div} T(x) - \left(\nabla T(x)\, \nu_E(x)\right) \cdot \nu_E(x)$$

$$= 0 - \left[(\nabla' \varphi(px), 0) \cdot \nu_E\right] \nu_E \cdot e_n$$

$$\therefore \quad \boxed{\operatorname{div}_E T(x) = -(\nu_E \cdot e_n)\left[(\nabla' \varphi(px), 0) \cdot \nu_E\right]} \quad (2)$$

We apply (2) in (1) to get:

$$\boxed{\int_M (\nu_E' \cdot \nabla' \varphi)(\nu_E \cdot e_n)\, d\mathcal{H}^{n-1} = 0} \quad (3)$$

We now define:

$$\Gamma_1 = M \cap \Gamma \cap \{\lambda = 1\} = \left\{ x \in M \cap \Gamma : \nu_E(x) = \frac{(-\nabla' u(px), 1)}{\sqrt{1 + |\nabla' u(px)|^2}} \right\},$$

which is the "good part" of $M \cap \Gamma$. We now show that a large portion of M is covered by Γ_1. Indeed, if $x \in (M \cap \Gamma) \setminus \Gamma_1$, then $\nu_E(x) \cdot e_n \leq 0$

and hence:

$$(4) \quad \boxed{e(E, 0, 1, e_n) = \int_M (1 - \nu_E \cdot e_n)\, d\mathcal{H}^{n-1} \geq \int_{(M \cap \Gamma) \setminus \Gamma_1} \overbrace{(1 - \nu_E \cdot e_n)}^{\geq 1} \geq \mathcal{H}^{n-1}((M \cap \Gamma) \setminus \Gamma_1).}$$

so that:

$$\mathcal{H}^{n-1}(M \Delta \Gamma_1) = \mathcal{H}^{n-1}(M \setminus \Gamma_1) + \mathcal{H}^{n-1}(\Gamma_1 \setminus M)$$

$$= \mathcal{H}^{n-1}(M \setminus \Gamma_1), \quad \text{since } \Gamma_1 \subset M$$

$\Rightarrow$

$$\mathcal{H}^{n-1}(M \Delta \Gamma_1) = \mathcal{H}^{n-1}(M \setminus \Gamma_1)$$ 30.8

$$\leq \mathcal{H}^{n-1}((M \cap \Gamma) \setminus \Gamma_1) + \mathcal{H}^{n-1}(M \setminus \Gamma)$$

Since:

$M \setminus \Gamma_1 \subset (M \cap \Gamma) \setminus \Gamma_1 + (M \setminus \Gamma)$; see picture below.

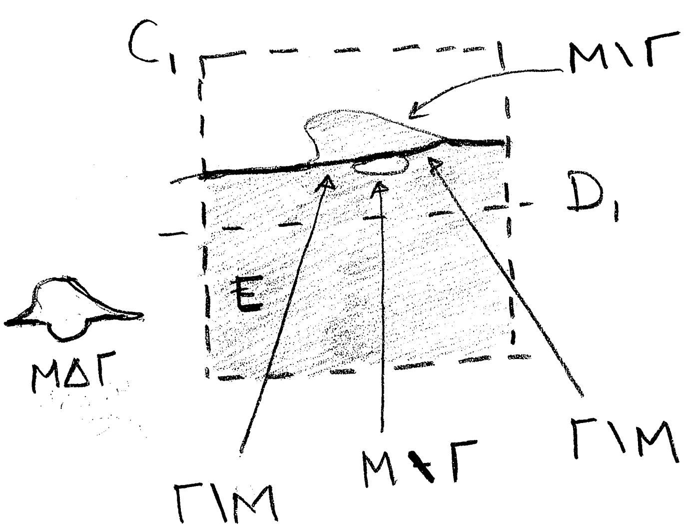
$$\Rightarrow \mathcal{H}^{n-1}(M \Delta \Gamma_1) \leq e(E, 0, 1, e_n) + \mathcal{H}^{n-1}(M \setminus \Gamma) \text{ ; by (4)}$$

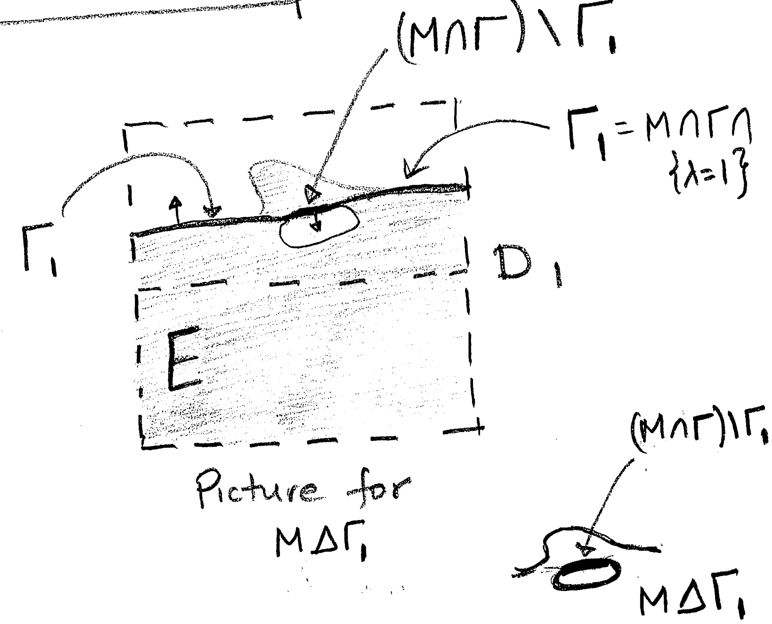
$$\leq 9^{n-1} e(E, 0, 9, e_n) + \mathcal{H}^{n-1}(M \setminus \Gamma)$$

$$\leq c(n) e(E, 0, 9, e_n) \text{ ;}$$ since, from previous Lecture: $\mathcal{H}^{n-1}(M \Delta \Gamma) \leq c(n) e(E, 0, 9, e_n)$.

We have proved

$$\boxed{\mathcal{H}^{n-1}(M \Delta \Gamma_1) \leq c(n) e(E, 0, 9, e_n)} \quad (5)$$





Picture for $M \Delta \Gamma_1$

Γ is darker boundary in picture
$M = \partial E \cap C_1$ is lighter boundary
Picture illustrating $M \Delta \Gamma$

By definition of Γ_1:

$$\int_{p(M\cap\Gamma_1)} \frac{\nabla' u \cdot \nabla'\varphi}{\sqrt{1+|\nabla' u|^2}} = \int_{M\cap\Gamma_1} \frac{\nabla' u(px)\cdot\nabla'\varphi(px)}{\sqrt{1+|\nabla' u(px)|^2}} (\nu_E\cdot e_n)\, d\mathcal{H}^{n-1}$$

$$= \int_{M\cap\Gamma_1} \frac{\nabla' u(px)\cdot\nabla'\varphi(px)}{1+|\nabla' u(px)|^2}\, d\mathcal{H}^{n-1}; \quad \text{since } \nu_E = \frac{(-\nabla' u(px), 1)}{\sqrt{1+|\nabla' u(px)|^2}} \text{ on } \Gamma_1$$

$$= \int_{M\cap\Gamma_1} \left(\frac{\nabla' u(px)\cdot\nabla'\varphi(px)}{\sqrt{1+|\nabla' u(px)|^2}}\right)\left(\frac{1}{\sqrt{1+|\nabla' u(px)|^2}}\right) d\mathcal{H}^{n-1}$$

$$= \int_{M\cap\Gamma_1} \left[-(\nabla'\varphi(px), 0)\cdot\nu_E\right]\left[\nu_E\cdot e_n\right] d\mathcal{H}^{n-1}$$

$$= \int_{M\cap\Gamma_1} \operatorname{div}_E T(x)\, d\mathcal{H}^{n-1} \quad ; \quad \text{see (2) and (3)}$$

$$= \int_{M\setminus\Gamma_1} \operatorname{div}_E T(x)\, d\mathcal{H}^{n-1}; \quad \text{because } \int_M \operatorname{div}_E T = 0 \;\; \text{and} \;\; \int_M \operatorname{div}_E T = \int_{M\cap\Gamma_1}\operatorname{div}_E T + \int_{M\setminus\Gamma_1}\operatorname{div}_E T$$

$$= \int_{M\setminus\Gamma_1} -(\nu_E\cdot e_n)\left[(\nabla'\varphi(px), 0)\cdot\nu_E\right] d\mathcal{H}^{n-1}$$

$$\therefore \left|\int_{p(M\cap\Gamma_1)} \frac{\nabla' u\cdot\nabla'\varphi}{\sqrt{1+|\nabla' u|^2}}\right| \le \sup_{D_1}|\nabla'\varphi|\,\mathcal{H}^{n-1}(M\setminus\Gamma_1)$$

$$\le \sup_{D_1}|\nabla'\varphi|\,\mathcal{H}^{n-1}(M\Delta\Gamma_1) \le C(n)\sup_{D_1}|\nabla'\varphi|\, e(E,0,9,e_n);$$

by (5).

Hence we have proved:

$$\boxed{\left| \int_{p(M\cap\Gamma_1)} \frac{\nabla' u \cdot \nabla' \varphi}{\sqrt{1+|\nabla' u|^2}} \right| \le C(n) \sup_{D_1} |\nabla' \varphi| \, e(E,0,9,e_n) \qquad (6)}$$

On the other hand:

$$\left| \int_{p(M\Delta\Gamma_1)} \frac{\nabla' u \cdot \nabla' \varphi}{\sqrt{1+|\nabla' u|^2}} \right| \le \sup_{D_1} |\nabla' \varphi| \, \mathcal{H}^{n-1}(p(M\Delta\Gamma_1))$$

$$\le \sup_{D_1} |\nabla' \varphi| \, \mathcal{H}^{n-1}(M\Delta\Gamma_1) \quad ; \quad \text{since } f \text{ Lip} \Rightarrow \mathcal{H}^s(f(E)) \le \operatorname{Lip}(f)^s \mathcal{H}^s(E).$$

$$\le C(n) \sup_{D_1} |\nabla' \varphi| \, e(E,0,9,e_n)$$

From these two inequalities, since $D_1 = p(M)$ we conclude:

$$\boxed{\left| \int_{D_1} \frac{\nabla' u \cdot \nabla' \varphi}{\sqrt{1+|\nabla' u|^2}} \right| \le 2C(n) \sup_{D_1} |\nabla' \varphi| \, e(E,0,9,e_n) \qquad (8)}$$

Finally show that (8) implies our desired inequality:

$$\left| \int_{D_1} \nabla u' \cdot \nabla' \varphi \right| \le C_1(n) \sup_{D_1} |\nabla' \varphi| \, e(E,0,9,e_n),$$

which concludes the proof of the Lipschitz approximation theorem.

Indeed, since $\mathrm{Lip}(u) \le 1$ and $\sqrt{1+s} \le 1 + \left(\frac{s}{2}\right)$, for every $s>0$:

$$\left|\int_{D_1} \nabla' u \cdot \nabla' \varphi\right| \le \left|\int_{D_1}\left(\frac{\nabla' u \cdot \nabla' \varphi}{\sqrt{1+|\nabla' u|^2}} - (\nabla' u \cdot \nabla' \varphi)\right)\right| + \left|\int_{D_1} \frac{\nabla' u \cdot \nabla' \varphi}{\sqrt{1+|\nabla' u|^2}}\right|$$

Note:
$$\left|\frac{\nabla' u \cdot \nabla' \varphi}{\sqrt{1+|\nabla' u|^2}} - (\nabla' u \cdot \nabla' \varphi)\right| = |\nabla' u \cdot \nabla' \varphi| \frac{\sqrt{1+|\nabla' u|^2} - 1}{\sqrt{1+|\nabla' u|^2}}$$

$$\le \frac{|\nabla' u \cdot \nabla' \varphi| \left[\left(1 + \frac{|\nabla' u|^2}{2}\right) - 1\right]}{\sqrt{1+|\nabla' u|^2}}$$

$$\le \frac{|\nabla' \varphi|\, |\nabla' u|^2}{2\sqrt{1+|\nabla' u|^2}} \;; \quad \text{since } |\nabla' u| \le 1$$

$$\le \left(\sup_{D_1} |\nabla' \varphi|\right) |\nabla' u|^2 \;; \quad \text{since } \frac{1}{2\sqrt{1+|\nabla' u|^2}} \le 1$$

Therefore:

$$\left|\int_{D_1} \nabla' u \cdot \nabla' \varphi\right| \le \sup_{D_1} |\nabla' \varphi| \int_{D_1} |\nabla' u|^2 + \left|\int_{D_1} \frac{\nabla' u \cdot \nabla' \varphi}{\sqrt{1+|\nabla' u|^2}}\right|$$

$$\le C(n) \sup_{D_1} |\nabla' \varphi|\, e(E,0,9,e_n) + C(n) \sup_{D_1} |\nabla' \varphi|\, e(E,0,9,e_n)$$

by (8) and (*)

$\Longrightarrow$

$$\left|\int_{D_1} \nabla' \varphi \cdot \nabla' u\right| \le C_1(n) \sup_{D_1} |\nabla' \varphi|\, e(E,0,9,e_n)$$