Lecture 31

The reverse Poincaré inequality.

Notice that, from the statement of the Lipschitz approximation theorem, we need to focus now on showing that:

"A small excess assumption at a given point, scale and direction implies the uniform smallness of the excess at every point inside a smaller cylinder, with respect to every sufficiently small scale, and with respect to the same direction".

Indeed, if we can prove the above statement, then we would be able to say that the good set M_0 in the Lipschitz approximation theorem actually coincides with the whole boundary of E in the smaller cylinder.

In order to prove such result, we need a reverse height bound, in which the excess is controlled through a sort of L^2-height.

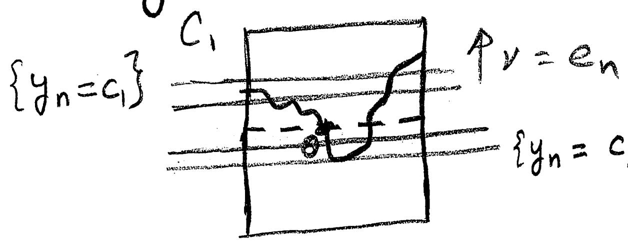
Precisely, we introduce the cylindrical flatness of a set of locally finite perimeter $E \subset \mathbb{R}^n$ at $x \in \mathbb{R}^n$ with respect to $\nu \in S^{n-1}$ at scale $r > 0$, as:

$$f(E, x, r, \nu) = \inf_{c \in \mathbb{R}} \frac{1}{r^{n-1}} \int_{C(x,r,\nu) \cap \partial^* E} \frac{|(y-x)\cdot\nu - c|^2}{r^2} \, d\mathcal{H}^{n-1}(y)$$

The flatness $f(E, x, r, \nu)$ measures the L^2-average distance of $\partial^* E$ to the family of hyperplanes

$$\{y : (y-x)\cdot\nu = c\}, \quad c \in \mathbb{R}$$

in the cylinder $C(x, r, \nu)$



$$f(E, 0, 1, e_n) = \inf_c \int_{C_1 \cap \partial^* E} |y_n - c|^2 \, d\mathcal{H}^{n-1}(y), \quad y = (y', y_n)$$

We have:

<u>Theorem (Reverse Poincaré inequality)</u>:

$\exists\, c(n)$ such that

If E minimizes perimeter in $C(x, 4r, \nu)$, $x \in \partial E$, and:

$$e(E, x, 4r, \nu) \le \omega(n, \tfrac{1}{8})$$

then:

$$\boxed{e(E, x, r, \nu) \le c(n)\, f(E, x, 2r, \nu) \qquad (*)}$$

Remark 1: The constant $w(n, \frac{1}{8})$ is the constant $w(n, t_0)$ that appears in the hypothesis of the "Small excess position Lemma".

Remark 2: The reverse Poincaré inequality is analogous to the Cacciopoli inequality, which plays a key role in the regularity theory for weak solutions of:

$$-\operatorname{div}(A(x)\nabla u(x)) = 0. \qquad (**)$$

The Cacciopoli inequality is:

Cacciopoli inequality: If $u \in W^{1,2}_{loc}(B)$ is a weak solution of the elliptic equation $(**)$, then

$$\int_{B(x,r/2)} |\nabla u|^2 \leq 16\left(\frac{\Lambda}{\lambda}\right)^2 \int_{B(x,r)} \frac{|u-(u)_{x,r}|^2}{r^2}$$

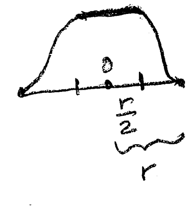
whenever $B(x,r) \subset\subset B$ and $(u)_{x,r} = \frac{1}{\omega_n r^n}\int_{B(x,r)} u$.

Proof: $\left.\begin{array}{l} (A x e)\cdot e \geq \lambda |e|^2 \ \forall e \in \mathbb{R}^n \\ \Lambda = \|A\|_{L^\infty} \end{array}\right\}$ Meaning of the constants Λ and λ in theorem

Up to a translation of u we may assume $(u)_{x,r} = 0$

Given $\varepsilon > 0$, let $\varphi \in C^\infty_c(B(x,r))$ such that:

$0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on $B(x, \frac{r}{2})$, $|\nabla \varphi| \leq \frac{2+\varepsilon}{r}$.



$\Rightarrow$ Define: (31.4)

$\Phi = \varphi^2 u \in W^{1,2}(B)$, $B = B(0,1)$

$\operatorname{Spt} \Phi \subset\subset B(x,r) \subset\subset B$

$$\nabla \Phi = 2u\varphi\nabla\varphi + \varphi^2 \nabla u$$

Since u is a weak solution $\Rightarrow$

$$\int_B A(x)\nabla u(x)\cdot\nabla\phi(x)\,dx = 0 \quad \forall \phi \in C_c^\infty(B)$$

Then, we can plug $\nabla\Phi$ in this expresion:

$$\int_B A(x)\nabla u(x)\cdot(2u\varphi\nabla\varphi + \varphi^2\nabla u)\,dx = 0$$

$\Rightarrow$

$$\lambda\int_B \varphi^2|\nabla u(x)|^2 dx \le \int_B \varphi^2 A(x)\nabla u(x)\cdot\nabla u(x)\,dx = -\int_B 2u\varphi A(x)\nabla u\cdot\nabla\varphi\,dx$$

$$\le 2\Lambda\int_B |u||\varphi|\,|\nabla u|\,|\nabla\varphi|\,dx$$

$$\le 2\Lambda\left(\int_B |\varphi|^2|\nabla u|^2\right)^{1/2}\left(\int_B |u|^2|\nabla\varphi|^2\right)^{1/2}$$

$$\Rightarrow \left(\int_B \varphi^2|\nabla u|^2 dx\right)^{1/2} \le 2\frac{\Lambda}{\lambda}\left(\int_B u^2|\nabla\varphi|^2\right)^{1/2}$$

$$\vee \!/$$

$$\left(\int_{B(x,\frac{r}{2})}|\nabla u|^2 dx\right)^{1/2} \qquad \le 2(2+\varepsilon)\frac{\Lambda}{\lambda}\left(\int_{B(x,r)}\frac{u^2}{r^2}\right)^{1/2}; \text{ since } |\nabla\varphi| \le \frac{2+\varepsilon}{r}$$

Hence, $$\int_{B(x,r/2)}|\nabla u|^2 dx \le 16\left(\frac{\Lambda}{\lambda}\right)^2\int_{B(x,r)}\frac{u^2}{r^2}\,dx \qquad \blacksquare$$

Remark 3: The main step in the proof of the reverse poincaré inequality is the construction of suitable comparison sets. The proof is long, but the ideas in the proof won't be needed later. Thus, we will come back to its proof after we complete the proof of the $C^{1,\gamma}$- regularity theorem.

We now recall that the hypothesis of the Lipschitz approximation theorem is to have a "uniform smallness of the excess". This property will be deduced from an iteration procedure, which uses the "excess improvement by tilting" property: for every sufficiently small $\alpha \in (0,1)$, if $e(E,x,r,\nu)$ is small enough depending on α and n, then there exists $\nu_0 \in S^{n-1}$ such that:

$$e(E,x,\alpha r,\nu_0) \leq c(n)\,\alpha^2 e(E,x,r,\nu).$$

The proof of this property requires two lemmas on harmonic functions, which we now prove. Recall that if v is harmonic in $B = B(0,1)$, then, by an application of the divergence theorem, the following mean value property holds: 31.6

$$v(x) = \fint_{\partial B(x,r)} v \, d\mathcal{H}^{n-1} = \fint_{B(x,r)} v \qquad \forall B(x,r) \subset\subset B$$

We have:

<u>Lemma 1</u>: If v is harmonic in B and:

$$w(x) = v(0) + \nabla v(0) \cdot x \quad , \; x \in B \, , \; B = B(0,1)$$

then:

$$\sup_{B(0,\alpha)} |v - w| \leq c(n) \, \alpha^2 \, \|\nabla v\|_{L^2(B)} \, ,$$

for every $\alpha \in (0, \frac{1}{2}]$. In particular:

$$\frac{1}{\omega_n \alpha^n} \int_{B(0,\alpha)} \frac{|v-w|^2}{\alpha^2} \leq c(n) \alpha^2 \int_B |\nabla v|^2 .$$

<u>Proof of Lemma 1</u>:

Let $e \in S^{n-1}$ and $|x| < \frac{1}{2}$, and $r < \frac{1}{4}$.

$e \cdot \nabla v$ is harmonic in B, since:

$$\Delta (e \cdot \nabla v) = e \cdot \nabla (\Delta v) = 0 \, ; \text{ since } \Delta v = 0 .$$

By the mean-value property:

$$|e \cdot \nabla v(x)| = \left| \fint_{B(x,r)} e \cdot \nabla v \right|$$

$$= \frac{c(n)}{r^n} \left| \int_{\partial B(x,r)} v \, (e \cdot \nu_{B(x,r)}) \, d\mathcal{H}^{n-1} \right| \, ; \quad \text{since: } \int_{B(x,r)} \frac{\partial v}{\partial x_i} = \int_{\partial B(x,r)} v \nu_i \, d\mathcal{H}^{n-1}$$

$$\leq \frac{C(n)}{r^n} \int_{\partial B(x,r)} |v(y)| \, d\mathcal{H}^{n-1}$$

$$= \frac{C(n)}{r^n} \int_{\partial B(x,r)} \left| \fint_{B(y,r)} v(z)\,dz \right| d\mathcal{H}^{n-1}(y)$$

$$\leq \frac{C(n)}{r^n} \int_{\partial B(x,r)} \frac{1}{|B(y,r)|} \int_{B(y,r)} |v(z)|\,dz \; d\mathcal{H}^{n-1}(y)$$

$$= \frac{C(n)}{r^{2n}} \int_{\partial B(x,r)} \int_{B(y,r)} |v(z)| \, dz \, d\mathcal{H}^{n-1}(y)$$

$$= \frac{C(n)}{r^{2n}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_{B(y,r)}(z) \, |v(z)| \, dz \, d\mu, \quad \mu = \mathcal{H}^{n-1} \llcorner \partial B(x,r)$$

Let $g(y,z) = \chi_{B(y,r)}(z)$

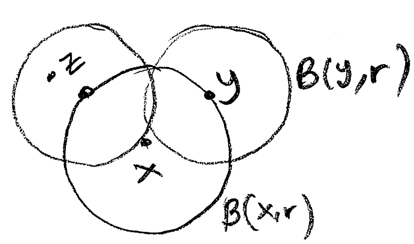
$$= \frac{C(n)}{r^{2n}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g(y,z)\,|v(z)|\,dz\,d\mu$$

$$= \frac{C(n)}{r^{2n}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g(y,z)\,|v(z)|\,d\mu\,dz;$$

By Fubini, which we can use since: $g(y,z)|v(z)| \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$

$$= \frac{C(n)}{r^{2n}} \int_{\mathbb{R}^n} |v(z)| \left(\int_{\mathbb{R}^n} g(y,z)\,d\mu \right) dz$$

Note that if $z \notin B(x,2r)$ then $g(y,z) = \chi_{B(y,r)}(z) = 0$



Hence:

$$|e\cdot\nabla v(x)| \le \frac{c(n)}{r^{2n}} \int_{B(x,2r)} |v(z)| \left(\int_{\mathbb{R}^n} g(y,z)\,d\mu \right) dz$$

$$= \frac{c(n)}{r^{2n}} \int_{B(x,2r)} |v(z)| \left(\int_{\partial B(x,r)} g(y,z)\,d\mathcal{H}^{n-1} \right) dz$$

But $g(y,z)=1$ for all $z\in B(x,2r)$. Indeed, given $z\in B(x,2r)$, then $z\in B(y,r)$, for some $y\in\partial B(x,r))$ and hence:

$$g(y,z) = \chi_{B(y,r)}(z) = 1$$; see picture in previous page.

$\therefore$

$$|e\cdot\nabla v(x)| \le \frac{c(n)}{r^{2n}} \int_{B(x,2r)} |v(z)|\, r^{n-1}\, dz$$

$$\le \frac{c(n)}{r^{2n}\cdot r^{1-n}} \int_{B(x,2r)} |v(z)|\,dz = \frac{c(n)}{r^{n+1}} \int_{B(x,2r)} |v(z)|\,dz.$$

We have proved:

$$\boxed{|e\cdot\nabla v(x)| \le \frac{c(n)}{r^{n+1}} \int_{B(x,2r)} |v(z)|\,dz} \quad (1)$$

If we apply (1) to $e = \dfrac{\nabla u(x)}{|\nabla u(x)|}$, $r=\frac{1}{8}$ $\Rightarrow$

$$|\nabla v(x)| \le c(n) \int_{B(0,1)} |v|\,dz$$; since $B(x,\frac{2}{8})\subset B(0,1)$ recall $x\in B(0,\frac{1}{2})$

Using Holder's inequality:

$$|\nabla v(x)| \le c(n)\left(\int_B |v|^2 dz\right)^{1/2} |B|^{1/2}$$

$$= c(n)\left(\int_B |v|^2 dz\right)^{1/2}$$

We have proved:

$$\boxed{\sup_{x\in B_{1/2}} |\nabla v| \le c(n)\, \|v\|_{L^2(B)}} \quad (2)$$

We now apply (2) to the harmonic function $e_i \cdot \nabla v$, $e_i = (0,\dots,1,\dots,0)$ yields:

$$\left|\left(\frac{\partial^2 v}{\partial x_i \partial x_1}, \frac{\partial^2 v}{\partial x_i \partial x_2}, \dots, \frac{\partial^2 v}{\partial x_i \partial x_n}\right)\right| \le c(n)\, \| e\cdot \nabla v\|_{L^2(B)}$$

$$\begin{matrix} \vee / \\ \left|\frac{\partial^2 v}{\partial x_i \partial x_j}\right|, \; j\in\{1,\dots,n\} \end{matrix} \qquad \le c(n)\, \|\nabla v\|_{L^2(B)}, \quad |x| < \frac{1}{2}$$

$$\Rightarrow \quad \boxed{\sup_{B_{1/2}} |D^2 v| \le c(n)\, \|\nabla v\|_{L^2(B)}} \quad (3)$$

where $D^2 v$ is the Hessian of v:

$$D^2 v = \begin{pmatrix} \frac{\partial^2 v}{\partial^2 x_1} & \frac{\partial^2 v}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 v}{\partial x_1 \partial x_n} \\ \frac{\partial^2 v}{\partial x_2 \partial x_1} & \frac{\partial^2 v}{\partial^2 x_n} & \cdots & \frac{\partial^2 v}{\partial x_2 \partial x_n} \\ \vdots & & & \\ \frac{\partial^2 v}{\partial x_n \partial x_1} & \frac{\partial^2 v}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 v}{\partial^2 x_n} \end{pmatrix}$$

By Taylor's formula:

$\forall x \in B$, $\exists t \in (0,1)$ such that:

$$v(x) = v(0) + \nabla v(0)\cdot x + R_1(x),$$

$$R_1(x) = \sum_{i,j=1}^{n} \frac{1}{2} \frac{\partial^2 v}{\partial x_i \partial x_j}(tx)x_i x_j, \quad x=(x_1,\dots,x_n)$$

$$\Rightarrow \boxed{|v(x)-w(x)| \leq C(n) \sup_{B_{1/2}} |D^2 v|\, |x|^2, \quad \forall x,\ |x|<\tfrac{1}{2}} \qquad (4)$$

Now, let $\alpha \in (0,\frac{1}{2}]$. If $|x|<\alpha$ we have from (4):

$$|v(x)-w(x)| \leq C(n) \sup_{B_{1/2}} |D^2 v|\, \alpha^2$$

$$\leq C(n)\alpha^2 \|\nabla v\|_{L^2(B)}, \quad \forall x,\ |x|<\alpha$$

Hence:

$$\boxed{\sup_{B(0,\alpha)} |v-w| \leq C(n)\alpha^2 \|\nabla v\|_{L^2(B)} \quad \forall \alpha \in (0,\tfrac{1}{2}]} \qquad (5)$$

Clearly, from (5):

$$\frac{1}{\omega_n \alpha^n}\int_{B(0,\alpha)} \frac{|v-w|^2}{\alpha^2} \leq \frac{1}{\cancel{\omega_n \alpha^n}} \frac{\left(\sup_{B(0,\alpha)}|v-w|\right)^2}{\alpha^2}\cdot \cancel{\omega_n \alpha^n}$$

$$\leq \frac{C(n)^2 \alpha^4}{\alpha^2} \|\nabla v\|_{L^2(B)}^2$$

$$= C(n)\,\alpha^2 \int_B |\nabla v|^2$$

$$\Rightarrow \boxed{\frac{1}{\omega_n \alpha^n}\int_{B(0,\alpha)} \frac{|v-w|^2}{\alpha^2} \leq C(n)\alpha^2 \int_{B_1} |\nabla v|^2} \qquad (6)$$

<u>Lemma 2</u> : (<u>Harmonic approximation</u>):

For every $\tau > 0$, there exists $\sigma > 0$ such that:

If $u \in W^{1,2}(B)$ satisfies:

$$\int_B |\nabla u|^2 \leq 1, \quad \left|\int_B \nabla u \cdot \nabla \varphi\right| \leq \sup_B |\nabla \varphi| \sigma \quad \forall \varphi \in C_c^\infty(B)$$

then, $\exists v$, harmonic on B such that:

$$\int_B |\nabla v|^2 \leq 1, \quad \int_B |v-u|^2 \leq \tau.$$

<u>Proof of Lemma 2</u> : We proceed by contradiction.

Thus $\exists \tau > 0$ and a sequence $\{u_k\}_{k=1}^\infty \subset W^{1,2}(B)$, such that:

$$\int_B |\nabla u_k|^2 \leq 1, \quad \left|\int_B \nabla u_k \cdot \nabla \varphi\right| \leq \frac{\|\nabla \varphi\|_{L^\infty(B)}}{k}, \quad \forall \varphi \in C_c^\infty(B)$$

but the conclusion is not true; that is:

$$(****) \quad \int_B |u_k - v|^2 \geq \tau > 0, \quad \forall v \text{ harmonic } \int_B |\nabla v|^2 \leq 1.$$

Consider:

$$w_k = u_k - c_k, \quad c_k = \fint_B u_k$$

$$\Rightarrow \|\nabla w_k\|_{L^2(B)} \leq 1, \quad \|w_k\|_{L^2(B)} = \|u_k - c_k\|_{L^2(B)}$$
$$\leq c(n) \|\nabla u_k\|_{L^2(B)}$$
$$\leq c(n) \text{; by Poincaré inequality}$$

$\Rightarrow \quad \|w_k\|_{W^{1,2}(B)} \leq C \quad \forall k$

$\Rightarrow$ Exists a subsequence, denoted again as $\{w_k\}$ and a function $w \in L^2(B)$ such that:

$$\boxed{w_k \to w \quad \text{in } L^2(B)} \; ;$$

recall that $W^{1,2}(B)$ is compactly embedded in $L^2(B)$.

Since $\|\nabla w_k\|_{L^2(B)} \leq 1$ then there exists $\tilde{w} \in L^2$ such that, up to a subsequence:

$$\nabla w_k \rightharpoonup \tilde{w} \, , \quad \text{weakly in } L^2.$$

<u>Claim 1</u>: $w \in W^{1,2}(B)$ and $\nabla w = \tilde{w}$.

Indeed, for any $\varphi \in C_c^\infty(B)$:

$$
\begin{aligned}
\langle \nabla w, \varphi \rangle &= -\int_B w \nabla \varphi \\
&= -\lim_{k \to \infty} \int_B w_k \nabla \varphi \; ; \quad \text{since } w_k \to w \text{ in } L^2(B) \\
&= \lim_{k \to \infty} \int_B \nabla w_k \, \varphi \; ; \quad \text{since } w_k \in W^{1,2}(B) \\
&= \int_B \tilde{w} \, \varphi \; ; \quad \text{since } \nabla w_k \rightharpoonup \tilde{w} \text{ in } L^2(B)
\end{aligned}
$$

$$\Rightarrow \quad \langle \nabla w, \varphi \rangle = \int_B \tilde{w} \varphi \quad \forall \varphi \in C_c^\infty(B)$$

$$\Rightarrow \quad \nabla w = \tilde{w} \in L^2(B) \Rightarrow w \in W^{1,2}(B).$$

Hence, we have:

$$\boxed{\begin{array}{ll} w_k \to w & \text{in } L^2(B) \\ \nabla w_k \to \nabla w & \text{in } L^2(B). \end{array}} \quad (7)$$

<u>Claim 2</u>: w is harmonic in B

Let $\varphi \in C_c^\infty(B)$. Then:

$$\left|\int_B \nabla w \cdot \nabla\varphi\right| \leq \left|\int_B \nabla(w-w_k)\cdot\nabla\varphi\right| + \left|\int_B \nabla w_k \cdot \nabla\varphi\right|$$

$$= \left|\int_B \nabla(w-w_k)\cdot\nabla\varphi\right| + \left|\int_B \nabla u_k \cdot \nabla\varphi\right|$$

$$\leq \left|\int_B \nabla(w-w_k)\cdot\nabla\varphi\right| + \frac{\|\nabla\varphi\|_{L^\infty(B)}}{k}$$

Letting $k\to\infty$ and using (7) yields:

$$\left|\int_B \nabla w\cdot\nabla\varphi\right| = 0 \quad \Rightarrow \quad \int_B \nabla w\cdot\nabla\varphi = 0 \quad \forall\varphi\in C_c^\infty(B)$$

$\Rightarrow$ w is harmonic in B; $\|\nabla w\|_{L^2(B)} \leq 1$.

Note now that <u>we get a contradiction because</u> since $w + c_k$ is harmonic $\forall k$ (c_k is a constant), by (****) we have:

$$I \leq \int_B |u_k - (w+c_k)|^2 = \int_B |u_k - c_k - w|^2 = \int_B |w_k - w|^2 \longrightarrow 0.$$

$\blacksquare$