Lecture 32

[W]e now prove the following [cr]ucial theorem:

Theorem (<u>Excess improvement by tilting</u>):

Given $\alpha \in (0, \frac{1}{72})$, $\exists\ \varepsilon_2(n,\alpha)$, $C_2(n)$ such that:
If
- E minimizes perimeter in $C(x,r,\nu)$, $x \in \partial E$
- $e(E, x, r, \nu) \le \varepsilon_2(n,\alpha)$

Then

$\exists\ \nu_0 \in S^{n-1}$ such that $e(E, x, \alpha r, \nu_0) \le C_2(n)\alpha^2 e(E, x, r, \nu)$

<u>Proof of the theorem</u>:

Up to replacing E with $E_{x, r/9} = \dfrac{E - x}{r/9}$ and up to a rotation taking ν into e_n we reduce the theorem to the following:

Given $\alpha \in (0, \frac{1}{72})$, $\exists\ \varepsilon_2(n,\alpha)$, $C_2(n)$ such that:
If
- E minimizes perimeter in $C_9 = C(0,9,e_n)$
- $e(E, 0, 9, e_n) \le \varepsilon_2(n,\alpha)$

Then

$\exists\ \nu_0 \in S^{n-1}$ such that $e(E, 0, 9\alpha, \nu_0) \le C_2(n)\alpha^2 e(E, 0, 9, e_n)$

Let $\varepsilon_0(n)$, $C_0(n)$, $\varepsilon_1(n)$ and $C_1(n)$ be the universal constants determined in the height bound thm and in the Lipschitz approximation theorem.

We now choose $\varepsilon_2(n,\alpha)$ small enough so that $\varepsilon_2(n,\alpha) \leq \min\{\varepsilon_0(n), \varepsilon_1(n)\}$. Note, however, that we imposed the condition $\varepsilon_1(n) \leq \varepsilon_0(n)$ in the proof of the Lipschitz approximation theorem.

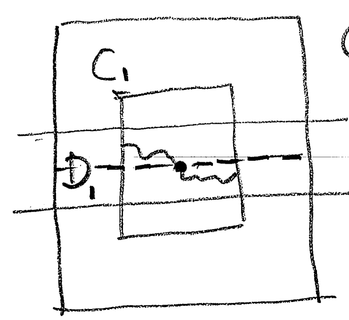
Thus, since both theorems apply, we inmediately have the following information:

$\exists\, u : \mathbb{R}^{n-1} \to \mathbb{R}$, Lipschitz, $\mathrm{Lip}(u) \leq 1$ such that

- $\displaystyle\sup_{\mathbb{R}^{n-1}} |u| \leq C_1(n)\, e(E,0,9,e_n)^{\frac{1}{2(n-1)}}$
- $\mathcal{H}^{n-1}(M \Delta \Gamma) \leq C_1(n)\, e(E,0,9,e_n)$, $\Gamma = \{(u, u(z)) : z \in D_1\}$

  Recall $C_1 = D_1 \times [-1,1]$
- $\displaystyle\int_{D_1} |\nabla' u|^2 \, d\mathcal{H}^{n-1} \leq C_1(n)\, e(E,0,9,e_n)$
- $\displaystyle\left|\int_{D_1} \nabla' u \cdot \nabla' \varphi\right| \leq C_1(n) \sup_{D_1} |\nabla' \varphi|\, e(E,0,9,e_n)$, $\forall \varphi \in C_c^1(D_1)$
- $\sup\{|\underbrace{q y}_{= y_n}| : y \in M\} \leq C_0(n)\, e(E,0,9,e_n)^{\frac{1}{2(n-1)}}$

$C_1 = D_1 \times [-1,1]$



C_9

Recall: $M = \partial E \cap C_1$

We now want to apply the <u>harmonic approximation</u> lemma, that we proved last class, to the almost harmonic function u.

Let:

$$\beta = C_1(n)e(E,0,9,e_n), \quad \boxed{u_0 = \frac{u}{\sqrt{\beta}}}$$

Then:

$u_0 \in W^{1,2}(D_1)$ and:

$$\boxed{\int_{D_1} |\nabla' u_0|^2 \leq 1, \quad \left| \int_{D_1} \nabla' u_0 \cdot \nabla' \varphi \right| \leq \|\nabla' \varphi\|_{L^\infty(D_1)} \sqrt{\beta}, \quad \forall \varphi \in C_c^1(D_1)}$$

By the harmonic approximation lemma, $\forall \tau > 0, \exists \sigma(\tau)$ such that if

$$\boxed{\sqrt{\beta} \leq \sigma(\tau)}$$

then $\exists v_0$, harmonic function on D_1 with:

$$\boxed{\int_{D_1} |\nabla' v_0|^2 \leq 1, \quad \int_{D_1} |v_0 - u_0|^2 \leq \tau}$$

Therefore, the function $v = \sqrt{\beta}\, v_0$ is harmonic on D_1 and:

$$\boxed{\int_{D_1} |\nabla' v|^2 \leq \beta, \quad \int_{D_1} |v - u|^2 \leq \tau\beta} \quad (1)$$

We now want to apply the Lemma 1 proved in previous class to the harmonic function v. Recall that this lemma estimates the distance between v and the tangent plane:

$$w(z) = v(0) + \nabla' v(0)\cdot z\,, \quad z \in D_1\,.$$

Note that $36\alpha < \frac{1}{2}$ and hence we can apply Lemma 1 to conclude:

$$\sup_{D(0,36\alpha)} |v-w| \le c(n)(36\alpha)^2 \|\nabla' v\|_{L^2(D_1)}$$

$$\le c(n)\alpha^2\sqrt{\beta}\,; \quad \text{since } \int_{D_1} |\nabla' v|^2 \le \beta$$

Note: recall our notation:
$D_1 = D(0,1)$
And, in general:
$D_{36\alpha} = D(0,36\alpha)$

Hence we have proved:

$$\boxed{\sup_{D_{36\alpha}} |v-w| \le c(n)\alpha^2\sqrt{\beta}} \qquad (2)$$

Therefore:

$$\frac{1}{\alpha^{n+1}}\int_{D_{36\alpha}} |u-w|^2 \le \frac{1}{\alpha^{n+1}}\left[\int_{D_{36\alpha}} (|u-v| + |v-w|)^2\right]$$

$$\le \frac{2}{\alpha^{n+1}}\left[\int_{D_{36\alpha}} |u-v|^2 + \int_{D_{36\alpha}} |v-w|^2\right]$$

$\Longrightarrow$

$$\frac{1}{\alpha^{n+1}}\int_{D_{36\alpha}}|u-w|^2 \le \frac{2}{\alpha^{n+1}}\left[\int_{D_1}(u-v)^2+\int_{D_{36\alpha}}|v-w|^2\right]$$

$$\le \frac{2}{\alpha^{n+1}}\left[\tau\beta+\int_{D_{36\alpha}}\left(\sup|v-w|\right)^2\right]; \text{ by (1)}$$

$$= C(n)\frac{2}{\alpha^{n+1}}\left[\tau\beta+\left(\sup_{D_{36\alpha}}|v-w|\right)^2\alpha^{n-1}\right]$$

$$\le \frac{C(n)}{\alpha^{n+1}}\left[\tau\beta+\alpha^4\beta\,\alpha^{n-1}\right]; \text{ by (2)}$$

$$= C(n)\left[\frac{\tau}{\alpha^{n+1}}+\frac{\alpha^{4+n-1}}{\alpha^{n+1}}\right]\beta$$

$\Longrightarrow$

$$\boxed{\frac{1}{\alpha^{n+1}}\int_{D_{36\alpha}}|u-w|^2 \le C(n)\left[\frac{\tau}{\alpha^{n+1}}+\alpha^2\right]\beta} \qquad (3)$$

We now choose $\tau=\alpha^{n+3}$. The constant $\sigma(\tau)=\sigma(\alpha^{n+3})$ is the one given by the harmonic approximation lemma. We choose $\varepsilon_2(n,\alpha)$ small enough so that:

$$\sqrt{C_1(n)\,\varepsilon_2(n,\alpha)} \le \sigma(\alpha^{n+3}).$$

Note that for (3) to be true we must have:

$$\sqrt{\beta} = \sqrt{C_1(n)\,e(E,0,9,e_n)} \le \sigma(\alpha^{n+3}),$$

which holds because $\sqrt{\beta}=\sqrt{C_1(n)e(E,0,9,e_n)} \le \sqrt{C_1(n)\,\varepsilon_2(n,\alpha)} \le \sigma(\alpha^{n+3})$.

Thus, from (3) with $\tau = \alpha^{n+3}$:

$$\frac{1}{\alpha^{n+1}} \int_{D_{36\alpha}} |u-w|^2 \leq c(n)\left[\frac{\alpha^{n+3}}{\alpha^{n+1}} + \alpha^2\right]\beta$$

$$= c(n)\alpha^2\beta$$

$$\Rightarrow \quad \boxed{\frac{1}{\alpha^{n+1}} \int_{D_{36\alpha}} |u-w|^2 \leq c(n)\alpha^2\beta} \quad (4)$$

Our desired inequality $e(E,0,9\alpha,\nu_0) \leq C_2(n)\alpha^2 e(E,0,9,e_n)$ will follow from (4) and the reversed Poincare inequality with ν_0 given by:

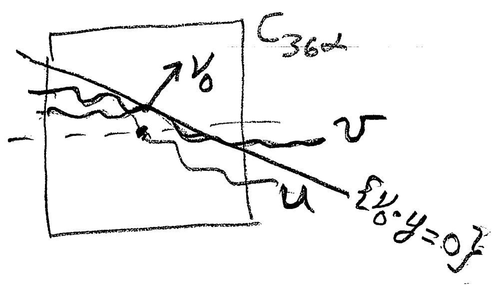
$$\boxed{\nu_0 = \frac{(-\nabla' v(0), 1)}{\sqrt{1+|\nabla' v(0)|^2}}} \quad (5)$$

We now proceed with the proof. The following claim is a key ingredient:

<u>Claim</u>: Assuming $e(E,0,9,e_n)$ even smaller so that $\beta^{\frac{1}{n-1}} \leq \alpha^{n+3}$ (recall that $\beta = C_1(n)e(E,0,9,e_n)$), we have:

$$\frac{1}{\alpha^{n+1}} \int_{M \cap C_{36\alpha}} |\nu_0 \cdot y - c|^2 \, d\mathcal{H}^{n-1}(y) \leq c(n)\alpha^2\beta$$

$$c = \frac{v(0)}{\sqrt{1+|\nabla' v(0)|^2}}$$



Recall, we want

$$e(E,0,9\alpha,\nu_0) \le C_2(n)\,\alpha^2 e(E,0,9,e_n)$$

We need $e(E,0,36\alpha,\nu_0) \le \omega(n,\frac{1}{8})$ so that we can apply the reverse poincaré inequality. Indeed, $\omega(n,\frac{1}{8})$ is the constant determined in the proof of the reverse Poincaré inequality. If this is true then:

$$e(E,0,9\alpha,\nu_0) \le C(n)\, f(E,0,18\alpha,\nu_0) \quad ; \text{ reverse Poincaré inequality}$$

$$\le C(n)2^{n-1} f(E,0,36\alpha,\nu_0) \quad ; \text{ property of cylindrical flat excess}$$

$$\le C(n)\frac{1}{\alpha^{n-1}}\int_{M\cap C_{36\alpha}} \frac{|\nu_0\cdot y - c|^2}{\alpha^2}\,d\mathcal{H}^{n-1}(y) \quad ; \text{ by definition of cylindrical flat excess}$$

$$= C(n)\frac{1}{\alpha^{n+1}}\int_{M\cap C_{36\alpha}} \frac{|\nu_0\cdot y - c|^2}{\alpha^2}\,d\mathcal{H}^{n-1}(y)$$

$$\le C(n)\alpha^2\beta \quad ; \text{ by our } \underline{\text{claim}}.$$

$$= C_2(n)\,\alpha^2 e(E,0,9,e_n)$$

which is our desired inequality:

$$\boxed{e(E,0,9\alpha,\nu_0) \le C_2(n)\,\alpha^2 (E,0,9,e_n)} \quad (*)$$

But we can choose $e(E,0,9,e_n)$ small enough, dependending on dimension (recall that we are choosing $\varepsilon_2(n,\alpha)$, $e(E,0,9,e_n) \le \varepsilon_2(n,\alpha)$) so that $e(E,0,36\alpha,\nu_0) \le \omega(n,\frac{1}{8})$. Indeed:

$$e(E,0,36\alpha,\nu_0) \le C(n)\left(e(E,0,36\sqrt{2}\alpha,e_n) + |\nu_0 - e_n|^2\right);$$

see Lecture 27 for properties of the excess when changing direction

$$\le C(n)\left[\left(\frac{9}{36\sqrt{2}\alpha}\right)^{n-1} e(E,0,9,e_n) + |\nu_0 - e_n|^2\right];$$

recall $s<r \Rightarrow$ $e(E,x,s,\nu) \le \left(\frac{r}{s}\right)^{n-1} e(E,x,r,\nu)$

$$= C(n)\,\frac{e(E,0,9,e_n)}{\alpha^{n-1}} + c(n)\cdot|\nu_0 - e_n|^2$$

$$\le c(n)\, e(E,0,9,e_n)^{1/n-1} + c(n)\,|\nu_0 - e_n|^2;$$

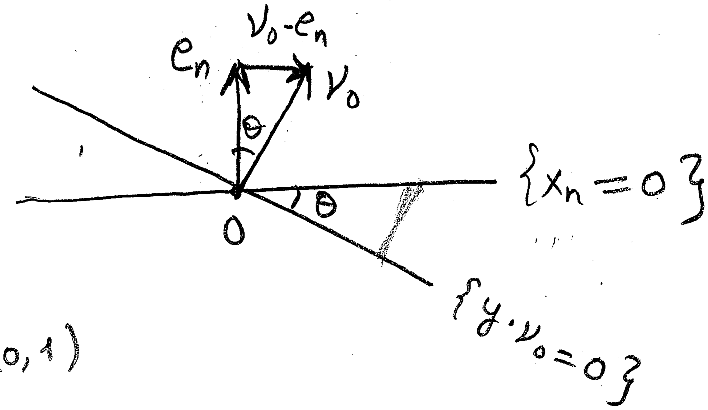
which is true if $e(E,0,9,e_n)$ is small enough so that:

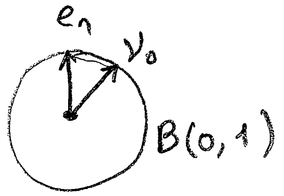
$$\frac{e(E,0,9,e_n)}{\alpha^{n-1}} \le e(E,0,9,e_n)^{\frac{1}{n-1}}$$

or $$e(E,0,9,e_n)^{\frac{n-2}{n-1}} \le \alpha^{n-1}.$$

or $$e(E,0,9,e_n) \le \alpha^{\frac{1}{n-2}}$$

Now, for the term $|\nu_0 - e_n|^2$





$$\nu_0 = \frac{(-\nabla' v(0), 1)}{\sqrt{1 + |\nabla' v(0)|^2}}$$

Clearly, $|\nu_0 - e_n|^2 = |2 - 2\nu_0 e_n| \le 4$. But, also:

$$
\begin{aligned}
|\nu_0 - e_n|^2 &\le c \tan\theta \\
&\le c\, |\nabla' v(0)|^2 \\
&\le c \left(\sup_{D_{1/2}} |\nabla' v|\right)^2 \\
&\le c(n) \int_{D_1} |v|^2 ;
\end{aligned}
$$

we proved last class that: v harmonic on $B \Rightarrow \sup_{B_{1/2}} |\nabla v| \le c(n) \|v\|_{L^2(B)}$

In this case $B = D_1$

$$\le c(n) \left(\int_{D_1} |v - u|^2 + \int_{D_1} u^2 \right)$$

$$\le c(n) \left(\alpha^{n+3} \beta + \int_{D_1} u^2 \right);$$ since $\int_{D_1} |v - u|^2 \le \tau \beta$ and we choose $\tau = \alpha^{n+3}$

$$\le c(n) \left(\alpha^{n+3} \beta + \beta^{1/n-1} \right);$$ since $\sup_{D_1} |u| \le c_1(n) e(E,0,9,e_n)^{\frac{1}{2(n-1)}}$

$$\le c(n) \left(\beta^{1/n-1} + \beta^{1/n-1} \right);$$ Choose $e(E,0,9,e_n)$ small enough so that $\alpha^{n+3}\beta \le \beta^{1/n-1}$

Therefore, since $\beta = c_1(n)\, e(E,0,9,e_n)$, we have:

$$e(E,0,36\alpha,\nu_0) \leq c(n)\, e(E,0,9,e_n)^{\frac{1}{n-1}}$$

and clearly, if $e(E,0,9,e_n)$ is small enough depending of dimension;

$$e(E,0,36\alpha,\nu_0) \leq \omega(n,\tfrac{1}{8}),$$

which makes possible to apply the reverse Poincaré inequality as in Page 32.7 to conclude the desired inequality (*).

Thus, the only thing left to prove is our <u>Claim</u> in page 32.6.

To prove the claim, we write;

$$M \cap C_{36\alpha} = (M \cap \Gamma \cap C_{36\alpha}) \cup [(M \setminus \Gamma) \cap C_{36\alpha}]$$

$\Rightarrow$

$$\frac{1}{\alpha^{n+1}} \int_{M\cap\Gamma\cap C_{36\alpha}} |\nu_0 \cdot y - c|^2\, d\mathcal{H}^{n-1}(y) = \frac{1}{\alpha^{n+1}} \int_{D_{36\alpha}} |\nu_0 \cdot (z,u(z)) - c|^2 \sqrt{1+|\nabla' u(z)|^2}\, dz,$$

since $\displaystyle\int_{\Gamma(u)} \varphi\, d\mathcal{H}^{n-1} = \int_{\mathbb{R}^{n-1}} \varphi(z,u(z))\sqrt{1+|\nabla' u(z)|^2}\, dz$, $\forall \varphi \in C_c(\mathbb{R}^n)$,

$u : \mathbb{R}^{n-1} \to \mathbb{R}$ Lipschitz

$\Gamma(u) = \{(z,u(z))\}$

Also:

$$\nu_0 \cdot (z, u(z)) = \frac{(-\nabla' v(0), 1)}{\sqrt{1+|\nabla' v(0)|^2}} \cdot (z, u(z))$$

$$= \frac{-z\nabla' v(0)}{\sqrt{1+|\nabla' v(0)|^2}} + \frac{u(z)}{\sqrt{1+|\nabla' v(0)|^2}}$$

$$\Rightarrow \nu_0 \cdot (z, u(z)) - c = \frac{-z\nabla' v(0)}{\sqrt{1+|\nabla' v(0)|^2}} + \frac{u(z)}{\sqrt{1+|\nabla' v(0)|^2}} - \frac{v(0)}{\sqrt{1+|\nabla' v(0)|^2}}$$

$$= \frac{u(z) - [v(0) + z\nabla' v(0)]}{\sqrt{1+|\nabla' v(0)|^2}}$$

$$= \frac{u(z) - w(z)}{\sqrt{1+|\nabla' v(0)|^2}}$$; $w(z)$ is the equation of tangent plane to the harmonic function $v(z)$ at $v(0)$.

$\therefore$

$$\frac{1}{\alpha^{n+1}} \int_{M\cap\Gamma\cap C_{36\alpha}} |\nu_0 \cdot y - c|^2 \, d\mathcal{H}^{n-1}(y) = \frac{1}{\alpha^{n+1}} \int_{D_{36\alpha}} \frac{|u(z)-w(z)|^2}{1+|\nabla' v(0)|^2} \cdot \sqrt{1+|\nabla' u(z)|^2} \, dz$$

$$\leq \frac{2}{\alpha^{n+1}} \int_{D_{36\alpha}} |u-w|^2 dz$$; since $\mathrm{Lip}(u) \leq 1$ $\Rightarrow |\nabla' u(z)|^2 \leq 1$

$$\leq C(n)\alpha^2\beta$$; by (4).

We have shown:

$$\boxed{\frac{1}{\alpha^{n+1}}\int_{M\cap\Gamma\cap C_{36\alpha}} |\nu_0\cdot y - c|^2 \, d\mathcal{H}^{n-1}(y) \le c(n)\alpha^2\beta} \quad (6)$$

We now compute the other piece of the integral, over $M\setminus\Gamma\cap C_{36\alpha}$.

$$\int_{(M\setminus\Gamma)\cap C_{36\alpha}} |\nu_0\cdot y - c|^2 \, d\mathcal{H}^{n-1}(y) = \int_{(M\setminus\Gamma)\cap C_{36\alpha}} |\nu_0\cdot(py, qy) - c|^2 \, d\mathcal{H}^{n-1}(y)$$

$$= \int_{(M\setminus\Gamma)\cap C_{36\alpha}} \left| \frac{-\nabla'v(0)\cdot py}{\sqrt{1+|\nabla'v(0)|^2}} + \frac{qy}{\sqrt{1+|\nabla'v(0)|^2}} - \frac{v(0)}{\sqrt{1+|\nabla'v(0)|^2}} \right|^2 d\mathcal{H}^{n-1}(y)$$

$$\le \int_{(M\setminus\Gamma)\cap C_{36\alpha}} |qy - (v(0) + \nabla'v(0)\cdot py)|^2 \, d\mathcal{H}^{n-1}(y); \quad \text{since } \frac{1}{1+|\nabla'v(0)|^2} \le 1$$

$$\le 2\int_{(M\setminus\Gamma)\cap C_{36\alpha}} |qy|^2 + |v(0) + \nabla'v(0)\cdot py|^2 \, d\mathcal{H}^{n-1}(y)$$

$$\le 4\,\mathcal{H}^{n-1}(M\setminus\Gamma)\left((\sup_{y\in M} |qy|)^2 + |v(0)|^2 + |\nabla'v(0)|^2 \right); \quad M = \partial E \cap C_1$$

$$\le C(n)\beta\left((\sup_{y\in M}|qy|)^2 + |v(0)|^2 + |\nabla'v(0)|^2 \right);$$

since $\mathcal{H}^{n-1}(M\Delta\Gamma) \le C_1(n)e(E,0,1,e_n)$ by the Lipschitz approximation thm. See Page 32.2

$\Rightarrow$

$$\int_{(M\setminus\Gamma)\cap C_{36\alpha}} |\nu_0\cdot y - c|^2 \, d\mathcal{H}^{n-1}(y) \le c(n)\beta\left(\beta^{\frac{1}{n-1}} + |v(0)|^2 + |\nabla' v(0)|^2\right); \quad (32.13)$$

Since $\sup_{y\in M} |q y| \le C_0(n)\, e(E,0,9,e_n)^{\frac{1}{2(n-1)}}$; by the height bound theorem; see Page 32.2. To estimate $|v(0)|^2 + |\nabla' v(0)|^2$ we proceed as in Page 32.9. Indeed:

$$|v(0)|^2 + |\nabla' v(0)|^2 \le c(n)\int_{D_1} |v|^2 + \left(\sup_{D_{1/2}} |\nabla' v|\right)^2;$$

by mean value property $\Rightarrow$ $v(0) = \fint_{D_1} v\, dz$

$$\le c(n)\int_{D_1} |v|^2,$$

and by Holder's inequality

and from here, as in Page 32.9, we conclude:

$$|v(0)|^2 + |\nabla' v(0)|^2 \le c(n)\left(\alpha^{n+3}\beta + \beta^{\frac{1}{n-1}}\right)$$

$\Rightarrow$

$$\int_{(M\setminus\Gamma)\cap C_{36\alpha}} |\nu_0\cdot y - c|^2 \, d\mathcal{H}^{n-1}(y) \le c(n)\beta\left(\beta^{\frac{1}{n-1}} + \alpha^{n+3}\beta + \beta^{\frac{1}{n-1}}\right)$$

$$\le c(n)\beta\left(\alpha^{n+3} + \alpha^{n+3}\beta + \alpha^{n+3}\right); \quad \text{since } \beta^{\frac{1}{n-1}} \le \alpha^{n+3}$$

$$\le c(n)\alpha^{n+3}\beta; \quad \text{since } \beta < 1.$$

$\Rightarrow$

$$\boxed{\frac{1}{\alpha^{n+1}}\int_{(M\setminus\Gamma)\cap C_{36\alpha}} |\nu_0\cdot y - c|^2 \, d\mathcal{H}^{n-1} \le c(n)\alpha^2\beta} \quad (7)$$

From (6) and (7) we conclude the <u>Claim</u>. ∎