Lecture 33

<u>Theorem</u> ($C^{1,\gamma}$-regularity theorem for local minimizers):

$\forall \gamma \in (0,1)$ $\exists\, \varepsilon_4(n,\gamma)$, $C_5(n,\gamma)$ s.t.:

<u>If</u>
- E minimizes perimeter in $C(x_0, 9r)$, $x_0 \in \partial E$
- $e(E, x_0, 9r, e_n) \leq \varepsilon_4(n,\gamma)$

<u>Then</u>:

$\exists u : \mathbb{R}^{n-1} \to \mathbb{R}$ Lipschitz with:

- $\sup_{\mathbb{R}^{n-1}} \frac{|u|}{r} \leq C_1(n)\, e(E, x_0, 9r, e_n)^{\frac{1}{2(n-1)}}$, $\quad \mathrm{Lip}(u) \leq 1$
- $C(x_0, r) \cap \partial E = x_0 + \{(z, u(z)) : z \in D_r\}$
- $C(x_0, r) \cap E = x_0 + \{(z,t) : z \in D_r, -r < t < u(z)\}$
- In fact $u \in C^{1,\gamma}(D(px_0, r))$, with
  - $|\nabla' u(z) - \nabla' u(z')| \leq C_5(n,\gamma)\, e(E, x, 9r, e_n)^{\frac{1}{2}} \left(\frac{|z-z'|}{r}\right)^{\gamma}$, $\forall z, z' \in D(px_0, r)$
  - $|\nu_E(x) - \nu_E(y)| \leq C_5(n,\gamma)\, e(E, x, 9r, e_n)^{1/2} \left(\frac{|x-y|}{r}\right)^{\gamma}$, $\forall x, y \in C(x_0, r) \cap \partial E$

<u>Note:</u> The proof of this theorem is based on the iteration of the "excess improvement by tilting":

$$e(E, x, \alpha r, \nu_0) \leq C_2(n)\, \alpha^2 e(E, x, r, \nu).$$

In order for this iteration to converge we need to replace $C_2(n)$ by 1 in this inequality. This can be done at the price of trading α^2 with the larger factor $\alpha^{2\gamma}$, $\gamma \in (0,1)$. This change is done in the following Lemma.

Lemma: $\forall \gamma(0,1)$, $\exists \alpha_0(n,\gamma) \in (0,1)$, $\varepsilon_3(n,\gamma)$, $C_3(n,\gamma)$ s.t.: (33.2)

If

- E minimizes perimeter in $C(x,r,\nu)$, $x \in \partial E$
- $e(E,x,r,\nu) \leq \varepsilon_3(n,\gamma)$

Then:

$\exists \nu_0 \in S^{n-1}$ such that:

- $e(E,x,\alpha_0 r,\nu_0) \leq \alpha_0^{2\gamma} e(E,x,r,\nu)$
- $|\nu_0 - \nu|^2 \leq C_3(n,\gamma) e(E,x,r,\nu)$

Proof of the Lemma:

Let $\alpha \in (0,\frac{1}{72})$, and let $C_2(n)$, $\varepsilon_2(n,\alpha)$ be the constants of the "excess improvement by tilting".

$$\gamma \in (0,1) \Rightarrow C_2(n)\alpha^2 = C_2(n)\alpha^{2(1-\gamma)}\alpha^{2\gamma} \leq \alpha^{2\gamma}, \quad \text{if } \alpha < C_2(n)^{\frac{-1}{2(1-\gamma)}}$$

Let $\alpha_0(n,\gamma)$ such that:

$$\alpha_0(n,\gamma) = \frac{1}{2}\min\left\{ \left(\frac{1}{C_2(n)}\right)^{\frac{1}{2(\gamma-1)}}, \frac{1}{72} \right\},$$

$$\varepsilon_3(n,\gamma) = \varepsilon_2(n,\alpha_0(n,\gamma))$$

We apply the "excess improvement by tilting" theorem with $\alpha = \alpha_0(n,\gamma) \in (0,\frac{1}{72})$. Hence, $\exists \nu_0$ s.t:

$$\begin{aligned} e(E,x,\alpha_0 r,\nu_0) &\leq C_2(n)\alpha_0^2 e(E,x,r,\nu) \\ &= C_2(n)\alpha_0^{2(1-\gamma)}\alpha_0^{2\gamma} e(E,x,r,\nu) \\ &\leq C_2(n) C_2(n)^{-1} \alpha_0^{2\gamma} e(E,x,r,\nu); \quad \text{since } \alpha_0 < C_2(n)^{\frac{-1}{2(1-\gamma)}} \\ &= \alpha_0^{2\gamma} e(E,x,r,\nu) \end{aligned}$$

$$\therefore \quad \boxed{e(E,x,\alpha_0 r,\nu_0) \leq \alpha_0^{2\gamma} e(E,x,r,\nu)} \quad (*)$$

Now:

$$|\nu_0-\nu|^2 \leq \left(|\nu_0-\nu_E| + |\nu_E-\nu|\right)^2$$
$$\leq 2|\nu_0-\nu_E|^2 + 2|\nu_E-\nu|^2$$

$$\Rightarrow \int_{C(x,\alpha_0 r,\nu_0)\cap\partial E} |\nu_0-\nu|^2 \, d\mathcal{H}^{n-1} \leq 2\int_{C(x,\alpha_0 r,\nu_0)\cap\partial E} |\nu_0-\nu_E|^2 \, d\mathcal{H}^{n+1} + 2\int_{C(x,\alpha_0 r,\nu_0)\cap\partial E} |\nu_E-\nu|^2 \, d\mathcal{H}^{n-1}$$

$$\Rightarrow \frac{1}{(\alpha_0 r)^{n-1}}\int_{C(x,\alpha_0 r,\nu_0)\cap\partial E} |\nu_0-\nu|^2 \, d\mathcal{H}^{n-1} \leq \underbrace{\frac{4}{(\alpha_0 r)^{n-1}}\int_{C(x,\alpha_0 r,\nu_0)\cap\partial E} \frac{|\nu_0-\nu_E|^2}{2} \, d\mathcal{H}^{n-1}}_{=\,4e(E,x,\alpha_0 r,\nu_0)} + \frac{2}{(\alpha_0 r)^{n-1}}\int_{C(x,\alpha_0 r,\nu_0)\cap\partial E} |\nu_E-\nu|^2 \, d\mathcal{H}^{n-1}$$

$$\geq \frac{|\nu_0-\nu|^2}{(\alpha_0 r)^{n-1}} P(E; C(x,\alpha_0 r,\nu_0))$$

$$\geq C(n)|\nu_0-\nu|^2 \text{; by}$$

the lower uniform densities for minimizers

$$\therefore\ C(n)|\nu_0-\nu|^2 \leq 4e(E,x,\alpha_0 r,\nu_0) + \frac{2}{(\alpha_0 r)^{n-1}}\int_{C(x,\alpha_0 r,\nu_0)\cap\partial E} |\nu_E-\nu|^2 \, d\mathcal{H}^{n-1}$$

$$\leq 4\alpha_0^{2\gamma} e(E,x,r,\nu) + \frac{2}{(\alpha_0 r)^{n-1}}\int_{C(x,\alpha_0 r,\nu_0)\cap\partial E} |\nu_E-\nu|^2 \, d\mathcal{H}^{n-1} \text{ ; by } (*)$$

$$\leq 4e(E,x,r,\nu) + \frac{2}{(\alpha_0 r)^{n-1}}\int_{C(x,\alpha_0 r,\nu_0)\cap\partial E} |\nu_E-\nu|^2 \, d\mathcal{H}^{n-1} \text{ ; } \alpha_0^{2\gamma} < 1$$

Notice that:

Since $\alpha_0 < \frac{1}{\sqrt{2}} \Rightarrow C(x, \alpha_0 r, \nu_0) \subset B(x_0, r) \subset C(x, r, \nu)$. Hence;

$$C(n)\,|\nu_0 - \nu|^2 \le 4\,e(E, x, r, \nu) + \frac{2}{(\alpha_0)^{n-1}} \cdot \frac{2}{r^{n-1}} \int_{C(x,r,\nu)\cap\partial E} \frac{|\nu_E - \nu|^2}{2}\, d\mathcal{H}^{n-1}$$

$$= 4\,e(E, x, r, \nu) + \frac{4}{(\alpha_0)^{n-1}}\, e(E, x, r, \nu)$$

$$\Rightarrow |\nu_0 - \nu|^2 \le \underbrace{\frac{4}{C(n)}\left(1 + \frac{1}{\alpha_0^{n-1}}\right)}_{C_3(n,\gamma)} e(E, x, r, \nu)$$

$$\Rightarrow \boxed{|\nu_0 - \nu|^2 \le C_3(n,\gamma)\, e(E, x, r, \nu)} \quad (**)$$

Proof of the $C^{1,\gamma}$-regularity theorem:

step one: We show that $C(x_0, r)\cap\partial E$ is actually the graph of a Lipschitz function. This is a consequence of the Lipschitz approximation theorem and the following claim:

Claim: Given $\gamma \in (0,1)$, let $\varepsilon_\varepsilon(n,\gamma)$ be the constant in Lemma.

If

- $e(E, x_0, 9r, e_n) \le \left(\frac{8}{9}\right)^{n-1} \varepsilon_3(n,\gamma)$

Then

$\forall x \in C(x_0, r)\cap\partial E$, $\exists \nu(x) \in S^{n-1}$ and $\exists C_4(n,\gamma)$ such that:

(a) $e(E, x, s, \nu(x)) \le C_4(n,\gamma)\left(\frac{s}{r}\right)^{2\gamma} e(E, x_0, 9r, e_n)$, $\forall s \in (0, 4r)$

(b) $|\nu(x) - e_n|^2 \le C_4\, e(E, x_0, 9r, e_n)$

(c) $e(E, x, s, e_n) \le C_4\, e(E, x_0, 9r, e_n)$ $\forall s \in (0, 8r)$.

Assume that the previous claim is true.

With $C_4(n,\gamma)$ from claim, we define:

$$\varepsilon_4(n,\gamma) = \min\left\{ \varepsilon_0(n),\ \varepsilon_1(n),\ \left(\frac{8}{9}\right)^{n-1}\varepsilon_3(n,\gamma),\ \frac{\delta_0(n)}{C_4(n,\gamma)} \right)$$

From Height bound theorem

From Lipschitz approximation theorem

From Lemma Page 33.2

From Lipschitz approxima-tion theorem

$$e(E,x_0,9r,e_n) \leq \varepsilon_4(n,\gamma) \Rightarrow e(E,x_0,9r,e_n) \leq \left(\frac{8}{9}\right)^{n-1}\varepsilon_3(n,\gamma) \Rightarrow \text{(a), (b), (c) hold in Claim}$$

Recall, we introduced in the Lipschitz approximation theorem the set:

$$M_0 = \left\{ x \in C(x_0,r) \cap \partial E : \sup_{0<s<8r} e(E,x,s,e_n) \leq \delta_0(n) \right\}$$

By (c) we have:

$$\boxed{M_0 = C(x_0,r) \cap \partial E}$$

Thus, since $e(E,x_0,9r,e_n) \leq \varepsilon_4(n,\gamma) \leq \varepsilon_1(n)$ we can apply the Lipschitz approximation theorem to get:

$\exists\, u : \mathbb{R}^{n-1} \to \mathbb{R}$ Lipschitz such that:

- $C(x_0,r) \cap \partial E \subset \Gamma = x_0 + \{(z,u(z)) : z \in \mathbb{R}^{n-1}\}$.
- $\displaystyle\sup_{\mathbb{R}^{n-1}} \frac{|u|}{r} \leq C_1(n)\, e(E,x_0,9r,e_n)^{\frac{1}{2(n-1)}}$

By the Lipschitz graph criterion (see Lecture 25) we have:

- $C(x_0,r) \cap \partial E = \{x_0 + \{(z,u(z)) : z \in D_r\}$

By the Small-excess position Lemma (see Lecture 27) we have:

- $C(x_0,r) \cap E = x_0 + \{(z,t) : z \in D_r,\ -r < t < u(z)\}$

and clearly now

- $\displaystyle \nu_E(x) = \frac{(-\nabla' u(px), 1)}{\sqrt{1+|\nabla' u(px)|^2}}, \quad \forall x \in C(x_0,r).$

Step two : Proof of the claim:

Fix $x \in C(x_0, r) \cap \partial E$

Let $t = 8r$. Since $C(x, 8r) \cap \partial E \subset [C(x_0, 9r) \cap \partial E]$ we have:

$$e(E, x, t, e_n) \leq \left(\frac{9}{8}\right)^{n-1} e(E, x_0, 9r, e_n)$$

$$\leq \left(\frac{9}{8}\right)^{n-1} \left(\frac{8}{9}\right)^{n-1} \varepsilon_3(n, \gamma) \text{; by hipothesis of Claim.}$$

$$= \varepsilon_3(n, \gamma)$$

$$\Rightarrow e(E, x, t, e_n) \leq \varepsilon_3(n, \gamma)$$

By Lemma, $\exists \nu_1 \in S^{n-1}$ such that:

first iteration $\begin{cases} e(E, x, \alpha t, \nu_1) \leq \alpha^{2\gamma} e(E, x, t, e_n), & \alpha = \alpha_0(n, \gamma) \\ |\nu_1 - e_n|^2 \leq C_3\, e(E, x, t, e_n), & C_3 = C_3(n, \gamma) \end{cases}$

We can apply the Lemma again because:

$$e(E, x, \alpha t, \nu_1) \leq e(E, x, t, e_n) \text{; since } \alpha^{2\gamma} < 1$$

$$\leq \varepsilon_3(n, \gamma)$$

By Lemma, $\exists \nu_2 \in S^{n-1}$ such that:

second iteration $\begin{cases} e(E, x, \alpha^2 t, \nu_2) \leq \alpha^{2\gamma} e(E, x, \alpha t, \nu_1) \\ |\nu_2 - \nu_1|^2 \leq C_3\, e(E, x, \alpha t, \nu_1) \end{cases}$

We can apply the Lemma again because:

$$e(E, x, \alpha^2 t, \nu_2) \leq e(E, x, \alpha t, \nu_1) \leq \varepsilon_3(n, \gamma)$$

By Lemma, $\exists \nu_3 \in S^{n-1}$ such that:

third iteration $\begin{cases} e(E, x, \alpha^3 t, \nu_3) \leq \alpha^{2\gamma} e(E, x, \alpha^2 t, \nu_2) \\ |\nu_3 - \nu_2|^2 \leq C_3\, e(E, x, \alpha^2 t, \nu_2) \end{cases}$

Also, notice that:

$$\begin{aligned} e(E, x, \alpha^3 t, \nu_3) &\le \alpha^{2\gamma} e(E, x, \alpha^2 t, \nu_2) \\ &\le \alpha^{2\gamma} \cdot \alpha^{2\gamma} e(E, x, \alpha t, \nu_1) \\ &\le \alpha^{2\gamma} \cdot \alpha^{2\gamma} \cdot \alpha^{2\gamma} e(E, x, t, e_n) \\ &= (\alpha^{2\gamma})^3 e(E, x, t, e_n) \end{aligned}$$

We continue with this iteration to obtain the existence of:

$$\{\nu_i(x)\} \subset S^{n-1}, \quad \text{such that:}$$

$$(\ast\ast\ast) \quad \begin{matrix} i\text{th} \\ \text{iteration} \\ \nu_0 = e_n \end{matrix} \begin{cases} e(E, x, \alpha^i t, \nu_i) \le (\alpha^{2\gamma})^i e(E, x, t, e_n) \\ |\nu_i - \nu_{i-1}|^2 \le C_3\, e(E, x, \alpha^{i-1} t, \nu_{i-1}) \\ \qquad\qquad\quad \le C_3 (\alpha^{2\gamma})^{i-1} e(E, x, t, e_n) \end{cases}$$

We now show that $\{\nu_i(x)\}$ is a Cauchy sequence:

If $j \ge i \ge 1 \Rightarrow$

$$\begin{aligned} |\nu_j - \nu_{i-1}| &\le \sum_{k=i}^{j} |\nu_k - \nu_{k-1}| \le \sum_{k=i}^{j} \sqrt{C_3 (\alpha^{2\gamma})^{k-1} e(E, x, t, e_n)} \\ &= \sqrt{C_3\, e(E, x, t, e_n)} \sum_{k=i}^{j} (\alpha^{\gamma})^{k-1} \\ &\le \sqrt{C_3\, e(E, x, t, e_n)} \left(\sum_{k=i}^{\infty} (\alpha^{\gamma})^{k-1} \right) \\ &= \sqrt{C_3\, e(E, x, t, e_n)} \left\{ (\alpha^{\gamma})^{i-1} + (\alpha^{\gamma})^{i} + (\alpha^{\gamma})^{i+1} + \cdots \right\} \\ &\le \varepsilon, \text{ for } i \text{ large enough.} \end{aligned}$$

$\therefore \exists \nu(x)$ s.t. $\nu_i(x) \to \nu(x)$

We have:

$$|\nu_j - \nu_{i-1}| \le \sqrt{c_3 e(E,x,t,e_n)}\,(\alpha^\gamma)^{i-1}\left(1+\alpha^\gamma+(\alpha^\gamma)^2+\dots\right) \quad (33.8)$$

$$= \sqrt{c_3 e(E,x,t,e_n)}\,\frac{(\alpha^\gamma)^{i-1}}{1-\alpha^\gamma}, \quad \alpha=\alpha_0(n,\gamma)$$

$$= \tilde{C}(n,\gamma)\sqrt{e(E,x,t,e_n)}\ ; \quad c_3=c_3(n,\gamma)$$

$$\le \tilde{C}(n,\gamma)\sqrt{\left(\frac{9}{8}\right)^{n-1} e(E,x_0,9r,e_n)}$$

$$\therefore \quad |\nu_j-\nu_{i-1}|^2 \le C(n,\gamma)\,e(E,x_0,9r,e_n) \quad \forall j,i,\ j\ge i\ge 1$$

In particular, with $i=1$, and letting $j\to\infty$ we get:

$$\boxed{|\nu(x)-e_n|^2 \le C(n,\gamma)\,e(E,x_0,9r,e_n)},$$

which proves (b) in the claim.

In order to prove (a), we let $s\in(0,4r)$. Since $s\in(0,\frac{t}{2})$ (recall $t=8r$), then $\exists i\ge 0$ such that:

$$\boxed{\alpha^{i+1}t \le 2s \le \alpha^i t}$$

By properties of the excess:

$$e(E,x,s,\nu(x)) \le c(n)\left(e(E,x,\sqrt{2}s,\nu_i(x)) + |\nu(x)-\nu_i(x)|^2\right)$$

$$\le c(n)\left\{\left(\frac{\alpha^i t}{\sqrt{2}s}\right)^{n-1} e(E,x,\alpha^i t,\nu_i(x)) + |\nu(x)-\nu_i(x)|^2\right\},$$

since $\sqrt{2}s < 2s \le \alpha^i t$.

Hence:

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$$e(E,x,s,\nu(x)) \leq c(n)\left\{\left(\frac{\alpha^i t}{s}\right)^{n-1} e(E,x,\alpha^i t,\nu_i(x)) + |\nu(x)-\nu_i(x)|^2\right\}$$

$$\left(\frac{\alpha^i t}{s}\right)^{n-1} e(E,x,\alpha^i t,\nu_i(x)) \leq \frac{c(n)}{\alpha^{n-1}} e(E,x,\alpha^i t,\nu_i(x)) \quad ; \text{ since } \alpha^{i+1}t \leq 2s \Rightarrow \frac{1}{2s} \leq \frac{1}{\alpha^{i+1}t}$$

$$\leq \frac{c(n)}{\alpha^{n-1}} (\alpha^{2\gamma})^i e(E,x,t,e_n) \quad ; \text{ by } (***)$$

$$= \frac{c(n)}{\alpha^{n-1+2\gamma}} (\alpha^{2\gamma})^{i+1} e(E,x,t,e_n)$$

$$= \frac{c(n)}{\alpha^{n-1+2\gamma}} (\alpha^{i+1})^{2\gamma} e(E,x,t,e_n)$$

$$\leq \frac{c(n)}{\alpha^{n-1+2\gamma}} \left(\frac{s}{t}\right)^{2\gamma} e(E,x,t,e_n) \quad ; \text{ since } \alpha^{i+1}t \leq 2s \Rightarrow \frac{\alpha^{i+1}}{2} \leq \frac{s}{t}$$

$$\leq \underbrace{\frac{c(n)}{\alpha^{n-1+2}} \left(\frac{9}{8}\right)^{n-1}}_{C_4(n,\gamma)} \left(\frac{s}{t}\right)^{2\gamma} e(E,x_0,9r,e_n).$$

For the second term, from Page 33.8 and $j \to \infty$: 33.10

$$\Rightarrow \quad |\nu(x) - \nu_i(x)|^2 \le \frac{C(n,\gamma)}{(1-\alpha^\gamma)^2} e(E, x, t, e_n) \, (\alpha^\gamma)^{2i-2}$$

$$= \frac{C(n,\gamma)}{(1-\alpha^\gamma)^2} (\alpha^{2i-2})^\gamma \, e(E, x, t, e_n)$$

$$= \underbrace{\frac{C(n,\gamma)(\alpha^{i-3})^\gamma}{(1-\alpha^\gamma)^2}}_{\tilde{C}(n,\gamma),} (\alpha^{i+1})^\gamma e(E, x, t, e_n)$$

recall $\alpha = \alpha_0(n,\gamma)$

$\Rightarrow$

$$|\nu(x) - \nu_i(x)|^2 \le \tilde{C}(n,\gamma)(\alpha^{i+1})^{2\gamma} e(E, x, t, e_n)$$

$$\le \tilde{C}(n,\gamma) \left(\frac{s}{t}\right)^{2\gamma} e(E, x, t, e_n) \; ; \quad \alpha^{i+1} t \le 2s \Rightarrow \frac{\alpha^{i+1}}{2} \le \frac{s}{t}$$

$$\le \left(\frac{9}{8}\right)^{n-1} \tilde{C}(n,\gamma) \left(\frac{s}{t}\right)^{2\gamma} e(E, x_0, 9r, e_n)$$

$$= C(n,\gamma) \left(\frac{s}{t}\right)^{2\gamma} e(E, x_0, 9r, e_n)$$

Therefore, we have proved, since $t = 8r$:

$$\boxed{e(E, x, s, \nu(x)) \le C(n,\gamma) \left(\frac{s}{r}\right)^{2\gamma} e(E, x_0, 9r, e_n) \quad \forall s \in \underset{(0,4r)}{\underset{\shortparallel}{\left(0, \tfrac{t}{2}\right)}}}$$

which is (a) in our claim.

We now proceed to show (c) in our claim:

Let $s \in (0,t) = (0,8r)$.

$$e(E,x,s,e_n) \le c(n)\left(e(E,x,\sqrt{2}s,\nu(x)) + |\nu(x)-e_n|^2\right)$$

$$\le c(n)\, e(E,x,\sqrt{2}s,\nu(x)) + c(n,\gamma)\, e(E,x_0,9r,e_n); \text{ by (b)}$$

If $s \in (0,\frac{t}{4}) \Rightarrow \sqrt{2}s < \frac{t}{2} \Rightarrow$ We can apply (a) to get:

$$e(E,x,\sqrt{2}s,\nu(x)) \le c(n,\gamma)\left(\frac{\sqrt{2}s}{t}\right)^{2\gamma} e(E,x_0,9r,e_n)$$

$$\le c(n,\gamma)\, e(E,x_0,9r,e_n); \text{ since } \frac{\sqrt{2}s}{t} < \frac{1}{2}$$

We have proved:

- If $s \in (0,\frac{t}{4})$ then $e(E,x,s,e_n) \le c(n,\gamma)\, e(E,x_0,9r,e_n)$

Now, if $s \in (\frac{t}{4},t) = (2r,8r)$, since $s < 9r$, we can directly estimate:

$$e(E,x,s,e_n) \le \left(\frac{9r}{s}\right)^{n-1} e(E,x_0,9r,e_n)$$

$$\le \left(\frac{9}{2}\right)^{n-1} e(E,x_0,9r,e_n); \quad \text{since } s \ge 2r \Rightarrow \frac{1}{s} \le \frac{1}{2r}$$

We have shown:

$e(E,x,s,e_n) \le c(n,\gamma)\, e(E,x_0,9r,e_n)$, $\forall s \in (0,8r)$, $\forall x \in C(x_0,r)$; which is (c) in claim. ∎