

Lecture 34

34.1

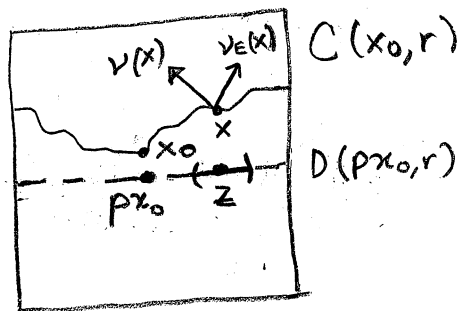
Continuation of the proof of the $C^{1,\alpha}$ -regularity theorem.

Step three: In order to show that $u \in C^{1,\alpha}(D(p_{x_0}, r))$ we will apply the Campanato's criterion (see Lecture 6). We now show that:

$$\frac{1}{s^{n-1}} \int_{D(z,s)} |\nabla u - (\nabla u)_{z,s}|^2 \leq C(n,\alpha) \left(\frac{s}{r}\right)^{2\alpha} e(E, x, q, r, e_n), \quad \forall z \in D(p_{x_0}, r) \text{ and } \forall s \in (0, r).$$

(*)

Fix $x \in C(x_0, r) \cap \partial E$
 $x = (z, u(z)).$



$x \in C(x_0, r) \cap \partial E$
 $x = (z, u(z)), \quad z \in D(p_{x_0}, r)$

$$\begin{aligned} \text{Write } v(x) &= (pv(x), qv(x)) \\ &= qv(x) \left(\frac{pv(x)}{qv(x)}, 1 \right) \end{aligned}$$

Let $\tau(x) = -\frac{pv(x)}{qv(x)}$, Note: $qv(x) \geq \frac{1}{\sqrt{2}}$ if the excess is small enough (see next page).

$$\Rightarrow v(x) = qv(x) (\tau(x), 1)$$

$$\Rightarrow 1 = qv(x) \sqrt{1 + |\tau(x)|^2} \Rightarrow qv(x) = \frac{1}{\sqrt{1 + |\tau(x)|^2}}$$

$$\text{And } pv(x) = qv(x) \cdot \frac{pv(x)}{qv(x)} = \frac{-\tau(x)}{\sqrt{1 + |\tau(x)|^2}} \Rightarrow v(x) = \left(\frac{-\tau(x)}{\sqrt{1 + |\tau(x)|^2}}, \frac{1}{\sqrt{1 + |\tau(x)|^2}} \right)$$

Note that, by choosing $\varepsilon_3(n, \delta)$ small enough, we have:

(34.2)

$$\boxed{q\nu(x) \geq \frac{1}{\sqrt{2}}} \quad (1)$$

Indeed, from the Claim proved in previous lecture, part (b), we have:

$$\boxed{|\nu(x) - e_n|^2 \leq C_4 e(E, x, q, r, e_n)} \quad (2)$$

So, if $e(E, x, q, r, e_n) \leq \left(\frac{8}{9}\right)^{n-1} \varepsilon_3(n, \delta)$ then the claim applies and we have (2), which implies (1).

Note that:

$$|\tau(x)| \leq 1,$$

because $|p\nu(x)|^2 + |q\nu(x)|^2 = 1$ and $|q\nu(x)|^2 \geq \frac{1}{2}$ implies that $|p\nu(x)|^2 \leq \frac{1}{2} \Rightarrow |p\nu(x)| \leq \frac{1}{\sqrt{2}}$. Hence,

$$|p\nu(x)| \leq q\nu(x) \Rightarrow \frac{|p\nu(x)|}{q\nu(x)} \leq 1 \Rightarrow |\tau(x)| \leq 1.$$

Notice now that, since $|\nabla u|^2 \leq 1$ due to the fact that $\text{Lip}(u) \leq 1$, the following two quantities are comparable:

$$(**) \quad \boxed{\frac{1}{\sqrt{1 + |\nabla u(w)|^2}} \sim \frac{1}{\sqrt{1 + |\tau(z, u(z))|^2}}} \quad , \quad \forall w \in D(z, s)$$

We can now compute (*):

$$\int_{D(z,s)} |\nabla' u - (\nabla' u)_{z,s}|^2 = \int_{D(z,s)} |\nabla' u - \tau(z, u(z)) + \tau(z, u(z)) - (\nabla' u)_{z,s}|^2 \quad (34.3)$$

$$\leq \int_{D(z,s)} (|\nabla' u(z) - \tau(z, u(z))| + |\tau(z, u(z)) - (\nabla' u)_{z,s}|)^2$$

$$\leq 2 \int_{D(z,s)} |\nabla' u(z) - \tau(z, u(z))|^2 + 2 \int_{D(z,s)} |\tau(z, u(z)) - (\nabla' u)_{z,s}|^2$$

$$= 2 \int_{D(z,s)} |\nabla' u - \tau(z, u(z))|^2 + 2C(n) |\tau(z, u(z)) - (\nabla' u)_{z,s}|^2$$

$$= 2 \int_{D(z,s)} |\nabla' u - \tau(z, u(z))|^2 + 2C(n) \left| \int_{D(z,s)} \tau(z, u(z)) - \int_{D(z,s)} \nabla' u \right|^2$$

$$= 2 \int_{D(z,s)} |\nabla' u - \tau(z, u(z))|^2 + 2\tilde{C}(n) \left(\int_{D(z,s)} |\tau(z, u(z)) - \nabla' u| \right)^2$$

$$\leq 2 \int_{D(z,s)} |\nabla' u - \tau(z, u(z))|^2 + \tilde{C}(n) \int_{D(z,s)} |\nabla' u - \tau(z, u(z))|^2 \quad ; \text{ by Hölder's inequality}$$

$$= C(n) \int_{D(z,s)} |\nabla' u - \tau(z, u(z))|^2 d\mathcal{H}^{n-1}$$

We have shown:

$$(3) \quad \int_{D(z,s)} |\nabla' u - (\nabla' u)_{z,s}|^2 d\mathcal{H}^{n-1} \leq c(n) \int_{D(z,s)} |\nabla' u - \tau(z, u(z))|^2 d\mathcal{H}^{n-1}$$

From (3) we have:

(34.4)

$$\int_{D(z,s)} |\nabla' u - (\nabla' u)_{z,s}|^2 \leq C(n) \int_{D(z,s)} |\nabla' u - \tau(z, u(z))|^2 \sqrt{1 + |\nabla' u|^2}$$

$$= C(n) (1 + |\tau(z, u(z))|^2) \int_{D(z,s)} \left| \frac{\nabla' u - \tau(z, u(z))}{\sqrt{1 + |\tau(z, u(z))|^2}} \right|^2 \sqrt{1 + |\nabla' u|^2}$$

$$\leq 2C(n) \int_{D(z,s)} \left| \frac{\nabla' u - \tau(z, u(z))}{\sqrt{1 + |\tau(z, u(z))|^2}} \right|^2 \sqrt{1 + |\nabla' u|^2} ; \text{ because } |\tau(z, u(z))|^2 \leq 1$$

$$= 2C(n) \int_{D(z,s)} \left| \frac{\nabla' u}{\sqrt{1 + |\tau(z, u(z))|^2}} - \frac{\tau(z, u(z))}{\sqrt{1 + |\tau(z, u(z))|^2}} \right|^2 \sqrt{1 + |\nabla' u|^2}$$

$$\leq C(n) \int_{D(z,s)} \left| \frac{\nabla' u}{\sqrt{1 + |\nabla' u|^2}} - \frac{\tau(z, u(z))}{\sqrt{1 + |\tau(z, u(z))|^2}} \right|^2 \sqrt{1 + |\nabla' u|^2} ; \text{ by } (**)$$

$$\leq C(n) \left(\int_{D(z,s)} \left| \frac{\nabla' u}{\sqrt{1 + |\nabla' u|^2}} - \frac{\tau(z, u(z))}{\sqrt{1 + |\tau(z, u(z))|^2}} \right|^2 \sqrt{1 + |\nabla' u|^2} \right. \\ \left. + \int_{D(z,s)} \left| \frac{1}{\sqrt{1 + |\nabla' u|^2}} - \frac{1}{\sqrt{1 + |\tau(z, u(z))|^2}} \right|^2 \sqrt{1 + |\nabla' u|^2} \right)$$

$$= C(n) \int_{D(z,s)} \left| \frac{(-\nabla' u, 1)}{\sqrt{1 + |\nabla' u|^2}} - \frac{(-\tau(z, u(z)), 1)}{\sqrt{1 + |\tau(z, u(z))|^2}} \right|^2 \sqrt{1 + |\nabla' u|^2}$$

We have shown:

$$\int_{D(z,s)} |\nabla' u - (\nabla' u)_{z,s}|^2 \leq C(n) \int_{D(z,s)} \left| \frac{(-\nabla' u, 1)}{\sqrt{1+|\nabla' u|^2}} - \frac{(-\tau(z, u(z)), 1)}{\sqrt{1+|\tau(z, u(z))|^2}} \right|^2 \sqrt{1+|\nabla' u|^2}$$

Recall that:

$$\bullet v_E(y) = \frac{(-\nabla' u(w), 1)}{\sqrt{1+|\nabla' u(w)|^2}}, \quad \forall y \in C(x,s)$$

$$y = (w, u(w))$$

$$\bullet v(x) = (p v(x), q v(x)) = \frac{(-\tau(x), 1)}{\sqrt{1+|\tau(x)|^2}}, \quad x = (z, u(z)).$$

and also, to pass the integral from $D(z,s)$ to $C(x,s) \cap \partial E$ we use:

$$\bullet \int_{\Gamma(u)} \varphi(y) d\mathcal{H}^{n-1}(y) = \int_{\mathbb{R}^{n-1}} \varphi(w, u(w)) \sqrt{1+|\nabla' u(w)|^2} dw.$$

Hence:

$$\int_{D(z,s)} |\nabla' u - (\nabla' u)_{z,s}|^2 \leq 2C(n) \int_{C(x,s) \cap \partial E} \left| \frac{v_E(y) - v(x)}{2} \right|^2 d\mathcal{H}^{n-1}(y)$$

$$\leq 2C(n) \int_{C(x,2s,v(x)) \cap \partial E} \left| \frac{v_E(y) - v(x)}{2} \right|^2 d\mathcal{H}^{n-1}(y),$$

since

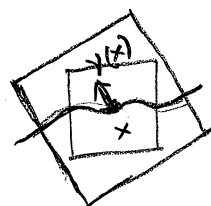
$$C(x,s) \subset C(x,2s,v(x))$$

because

$$qv(x) \geq \frac{1}{\sqrt{2}}$$

$$= C(n) (2s)^{n-1} \frac{1}{(2s)^{n-1}} \int_{C(x,2s,v(x)) \cap \partial E} \left| \frac{v_E(y) - v(x)}{2} \right|^2 d\mathcal{H}^{n-1}(y)$$

$$= C(n) s^{n-1} e(E, x, 2s, v(x))$$



But we are under the hypothesis of the claim proved in previous lecture:

34.6

$$e(E, x_0, 9r, e_n) \leq \left(\frac{8}{9}\right)^{n-1} \varepsilon_3(n, \delta)$$

and hence, from part (a) in that claim we obtain:

$$e(E, x, 2s, \nu(x)) \leq C_4(n, \delta) \left(\frac{s}{r}\right)^{2\delta} e(E, x_0, 9r, e_n); \quad \text{recall } 0 < s < r$$

$\Rightarrow 2s < 2r < 4r$
so (a) applies to this scale.

Therefore:

$$\left(\frac{1}{s^{n-1}} \int_{D(z, s)} |\nabla' u - (\nabla' u)_{z, s}|^2 \right)^{1/2} \leq \sqrt{C(n) e(E, x, 2s, \nu(x))} \\ \leq C(n, \delta) \left(\frac{s}{r}\right)^{\delta} \sqrt{e(E, x_0, 9r, e_n)}, \text{ which is } (*)$$

We can now apply Campanato's criterion to the function $\nabla' u \in L^2(D(p x_0, r))$. Then \exists a function $\overline{\nabla' u}: D(p x_0, r) \rightarrow \mathbb{R}$ with $\overline{\nabla' u} = \nabla' u$ for \mathcal{H}^{n-1} -a.e. $z \in D(p x_0, r)$ such that:

$$|\overline{\nabla' u}(z) - \overline{\nabla' u}(z')| \leq C_5(n, \delta) \sqrt{e(E, x_0, 9r, e_n)} \frac{|z - z'|^\delta}{r^\delta},$$

and we have proved that $u \in C^{1, \delta}(D(p x_0, r))$.

Finally, since

$$v \in \mathbb{R}^{n-1} \mapsto \frac{(-v, 1)}{\sqrt{1+|v|^2}} \in \mathbb{R}^n$$

defines a Lipschitz map on \mathbb{R}^{n-1} , then
for every $x, y \in C(x_0, r) \cap \partial E$ we have:

$$\begin{aligned} |v_E(x) - v_E(y)| &= \left| \frac{(-\nabla' u(p_x), 1)}{\sqrt{1+|\nabla' u(p_x)|^2}} - \frac{(-\nabla' u(p_y), 1)}{\sqrt{1+|\nabla' u(p_y)|^2}} \right| \\ &\leq C |\nabla' u(p_x) - \nabla' u(p_y)| \\ &\leq C(n, \sigma) \sqrt{e(E, x_0, r, e_n)} \left(\frac{|p_x - p_y|}{r} \right)^\sigma \\ &\leq C(n, \sigma) \sqrt{e(E, x_0, r, e_n)} \left(\frac{|x - y|}{r} \right)^\sigma, \end{aligned}$$

which completes the proof of the
Lipschitz / $C^{1, \sigma}$ regularity theorem for perimeter
minimizers.