Lecture 35

$C^{1,\gamma}$ regularity of the reduced boundary, and the characterization of the singular set.

The following theorem provides a useful characterization of the singular set of a perimeter minimizer.

<u>Theorem</u> (Regularity of the reduced boundary and the singular set). Let $A \subset \mathbb{R}^n$ open set, $n \geq 2$ and E a perimeter minimizer in A. Then:

- $A \cap \partial^* E$ is a $C^{1,\gamma}$-hypersurface for every $\gamma \in (0,1)$.
- $A \cap \partial^* E$ is relatively open in $A \cap \partial E$
- $\mathcal{H}^{n-1}(A \cap (\partial E \setminus \partial^* E)) = 0$
- $\exists\, \varepsilon(n)$ such that:

$$A \cap (\partial E \setminus \partial^* E) = \{x \in \partial E \cap A : e(E,x,r) \geq \varepsilon(n) \quad \forall r>0 \text{ s.t } B(x,r) \subset\subset A\}$$

<u>Remark 1</u>: Recall that $e(E,x,r)$ denotes the spherical excess;

$$e(E,x,r) = \min_{\nu \in S^{n-1}} \frac{1}{r^{n-1}} \int_{B(x,r) \cap \partial^* E} \frac{|\nu_E(y) - \nu|^2}{2} \, d\mathcal{H}^{n-1}(y)$$

<u>Remark 2</u>: We will use the notation:

$$\Sigma(E;A) = A \cap (\partial E \setminus \partial^* E)$$

Proof of the theorem:

If $\{\varepsilon_4(n,\gamma)\}_{0<\gamma<1}$ are the constants appearing in the theorem for the $C^{1,\gamma}$ regularity for local minimizers, we define:

$$\varepsilon(n) := \sup_{0<\gamma<1} \varepsilon_4(n,\gamma)$$

Define the set:

$$S = \{x \in A \cap \partial E : e(E,x,r) \geq \varepsilon(n) \ \forall r>0,\ B(x,r) \subset\subset A\}$$

We proved in Lecture 26 that:

$$\boxed{\lim_{r\to 0^+} e(E,x,r) = 0 \quad \forall x \in A \cap \partial^* E} \qquad (1)$$

From (1) we have:

$$\boxed{S \subset A \cap (\partial E \setminus \partial^* E)} \qquad (2)$$

Conversely, to see that $A \cap (\partial E \setminus \partial^* E) \subset S$ we take $x \in (A \cap \partial E) \setminus S$. We need to show that $x \in \partial^* E$.

Now:

$x \notin S \Rightarrow \exists\, r,\ B(x,r) \subset\subset A$ and $\gamma \in (0,1)$ such that:

$$e(E,x,9r) < \varepsilon_4(n,\gamma)$$

By definition of spherical excess, $\exists\, \nu \in S^{n-1}$ such that:

$$\boxed{e(E,x,9r,\nu) < \varepsilon_4(n,\gamma)} \qquad (3)$$

$C(x,9r,\nu)$

Note that (3) is precisely the hypothesis of the $C^{1,\gamma}$-regularity for local minimizers theorem. Hence, we infer from this theorem and (3) that:

$$\boxed{\begin{array}{l} C(x,r,\nu)\cap\partial E \text{ is the } (n-1)\text{-dimensional} \\ \text{graph of a function } u \text{ of class } C^{1,\gamma}. \end{array}} \quad (4)$$

From (4) it follows that $C(x,r,\nu)\cap\partial E = C(x,r,\nu)\cap\partial^* E$ and:

$$x\in\partial^* E,$$

which yields the other inclusion:

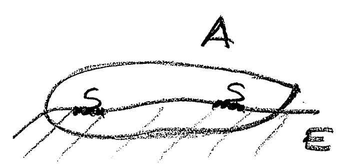
$$\boxed{A\cap(\partial E\setminus\partial^* E)\subset S} \quad (5)$$

From (2) and (5) we conclude that $S = A\cap(\partial E\setminus\partial^* E)$

Now, we have proved in a previous lecture that $\mathcal{H}^{n-1}(A\cap(\partial E\setminus\partial^* E)) = 0$; as a consequence of the uniform density estimates for minimizers.

Since the set S is closed in $\mathbb{R}^n$, then $\mathbb{R}^n\setminus S$ is open $\mathbb{R}^n$. And hence

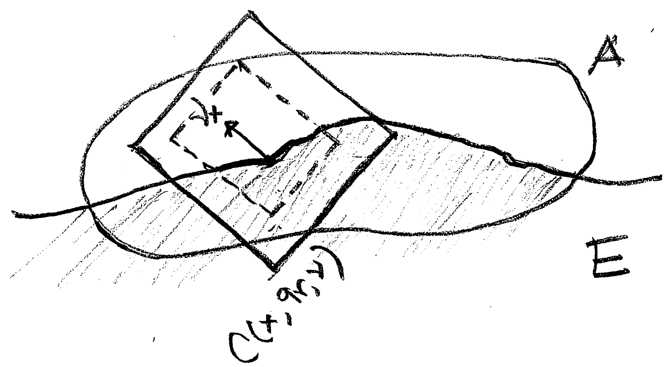
$$A\cap\partial^* E = (\mathbb{R}^n\setminus S)\cap(A\cap\partial E);$$



which means that $A\cap\partial^* E$ is relatively open in $A\cap\partial E$.

We now see clearly that $A \cap \partial^* E$ is a $C^{1,\gamma}$-hypersurface, because for every $x \in \partial^* E$, since $\lim_{r \to 0} e(E, x, r) = 0$ then $\exists\, \nu_x$ and $r > 0$ such that:

$$e(E, x, 9r, \nu_x) < \varepsilon_4(n, \gamma), \qquad (6)$$



Thus, from (6) we have that $C(x, r, \nu) \cap \partial E$ is the $(n-1)$-dimensional graph of a function of class $C^{1,\gamma}$. ∎

C^1-convergence for sequences of perimeter minimizers.

As a further application of the theorem for $C^{1,\gamma}$-regularity of local minimizers we show that the convergence of regular points of perimeter minimizers to a regular point of the limit set forces the convergence of the corresponding outer unit normals.

Theorem: Convergence of outer unit normals:

If $\{E_k\}$ and E are perimeter minimizers in the open set $A \subset \mathbb{R}^n$ and:

$$E_k \to E \text{ in } L^1_{loc}, \quad x_k \in A \cap \partial E_k, \quad x \in A \cap \partial^* E, \quad x_k \to x$$

then, for k large enough:

$$x_k \in A \cap \partial^* E_k \quad \text{and} \quad \lim_{k \to \infty} \nu_{E_k}(x_k) = \nu_E(x)$$

Proof:

Since $x_k \to x$, for k large enough, we can replace E_k with $E_k + (x - x_k)$ and A $A_\delta = \{x \in A : d(x, \partial A) > \delta\}$ for δ small enough and thus we may directly assume that $x_k = x$ $\forall k$.

$x \in \partial^* E \Rightarrow \exists r > 0, \nu \in S^{n-1}$ such that $C(x, 9r, \nu) \subset\subset A$ and:

$$e(E, x, 9r, \nu) < \varepsilon_4(n, \gamma), \quad \mathcal{H}^{n-1}(\partial^* E \cap \partial C(x, r, \nu)) = 0$$

Up to a common rotation of E and of all the E_k, we may assume $\nu = e_n$. Thus:

$$\boxed{e(E, x, 9r, e_n) < \varepsilon_4(n, \gamma), \quad \mathcal{H}^{n-1}(\partial^* E \cap \partial C(x, r)) = 0}$$

Since $e(E_k, x, 9r, e_n) \longrightarrow e(E, x, 9r, e_n)$, 35.6

(see Lecture 27), we have:

$$e(E_k, x, 9r, e_n) < \varepsilon_4(n, \gamma), \text{ for } k \text{ large enough}$$

$\Rightarrow e(E_k, x, 9r, e_n) < \varepsilon(n)$; $\varepsilon(n)$ is from previous theorem

$\Rightarrow e(E_k, x, 9r) < \varepsilon(n)$

$\Rightarrow x \notin \{x \in A \cap \partial E_k : e(E, x, r) \geq \varepsilon(n) \ \forall r > 0, B(x,r) \subset\subset A\}$

$\Rightarrow x \in \partial^* E_k \cap A$, for k large enough

Then, by the local regularity theorem,

$\exists u, u_k : D(px, r) \longrightarrow \mathbb{R}$, $\mathrm{Lip}(u), \mathrm{Lip}(u_k) \leq 1$ s.t:

- $C(x,r) \cap E = \{(z,t) : z \in D(px,r), -r < t < u(z)\}$
- $C(x,r) \cap E_k = \{(z,t) : z \in D(px,r), -r < t < u_k(z)\}$
- $|\nabla' u_k(z) - \nabla' u_k(z')| \leq C(n,\gamma)\left(\frac{|z-z'|}{r}^{\gamma}\right)$, $\forall z, z' \in D(px, r)$

Then:

$$\int_{D(px,r)} |u_k - u| = |(E_k \Delta E) \cap C(x,r)| \longrightarrow 0 \text{ as } k \to \infty.$$

$$\Rightarrow \int_{D(px,r)} \varphi \nabla' u_k = -\int_{D(px,r)} u_k \nabla' \varphi \longrightarrow -\int_{D(px,r)} u \nabla' \varphi = \int_{D(px,r)} \varphi \nabla u, \ \forall \varphi \in C^1_c(D(px,r))$$

Then $\{\nabla u'_k\}_{k=1}^{\infty}$ is equicontinuous

$\{\nabla u'_k\}$ is bounded $(\mathrm{Lip}(u_k) \leq 1)$.

$\Rightarrow$ By Ascoli-Arzelá theorem $\exists v$ s.t:

$\nabla u'_k \to v$ uniformly on $D(px, r)$

But:

$$\int \varphi \nabla u'_k \to \underset{\displaystyle \int \varphi \nabla u'}{\int \varphi v \atop \|} \quad \forall \varphi \Rightarrow \nabla u' = v \text{ on } D(px, r)$$

$\therefore$ $\boxed{\nabla u'_k \to \nabla u' \text{ uniformly on } D(px, r)}$ $(*)$

Recall that:

$$(**) \begin{cases} \nu_{E_k}(x_k) = \dfrac{(-\nabla u'_k(px_k), 1)}{\sqrt{1 + |\nabla u'_k(px_k)|^2}}, \\[2ex] \nu_E(x) = \dfrac{(-\nabla u'(px), 1)}{\sqrt{1 + |\nabla u'(px)|^2}} \end{cases}$$

From $(*)$ and $(**)$ we conclude that:

$\nu_{E_k}(x_k) \to \nu_E(x)$. ∎

Higher Regularity

Consider the area integral:

$f(w) = \sqrt{1+|w|^2}$

We now compute $D^2 f(w) = M(w)$.

$$\frac{\partial f}{\partial w_i} = \frac{1}{\cancel{2}}(1+|w|^2)^{-1/2}\cancel{2}w_i = \frac{w_i}{\sqrt{1+|w|^2}}, \quad i = 1, 2, \ldots$$

$$\frac{\partial^2 f}{\partial w_j \partial w_i} = \frac{\sqrt{1+|w|^2}\,\delta_{ij} - w_i\left(\frac{1}{\cancel{2}}\right)(1+|w|^2)^{-1/2}(\cancel{2}w_j)}{1+|w|^2}, \quad \delta_{ij} = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

$$= \frac{\sqrt{1+|w|^2}\,\delta_{ij} - \dfrac{w_i w_j}{\sqrt{1+|w|^2}}}{1+|w|^2}$$

$$= \frac{(1+|w|^2)\,\delta_{ij} - w_i w_j}{\sqrt{1+|w|^2}\,(1+|w|^2)}$$

$$= \frac{1}{\sqrt{1+|w|^2}}\left(\delta_{ij} - \frac{w_i w_j}{1+|w|^2}\right)$$

So $M = \dfrac{\partial^2 f}{\partial w_j \partial w_i}$ $1 \le i, j \le n$ is an $n \times n$ matrix.

The linear map M can be written as:

$$\boxed{M(w) = \frac{1}{\sqrt{1+|w|^2}}\left(Id - \frac{w \otimes w}{1+|w|^2}\right), \quad w \in \mathbb{R}^n} \qquad (7)$$

Indeed;

If $v = v_1 e_1 + \cdots + v_n e_n$ then we only need to see they agree on each e_i $i = 1, 2, \ldots, n$. First,

$$\Rightarrow \frac{1}{\sqrt{1+|w|^2}}\left(Id - \frac{w \otimes w}{1+|w|^2}\right) e_i = \frac{1}{\sqrt{1+|w|^2}}\left(e_i - \frac{(w \cdot e_i) w}{1+|w|^2}\right)$$

$$= \frac{1}{\sqrt{1+|w|^2}}\left(e_i - \frac{w_i}{1+|w|^2} w\right)$$

On the other, working with the matrix $\left(\frac{\partial^2 f}{\partial w_j \partial w_i}\right)$, $1 \le i, j \le n$; we have:

$$[M(w)] e_i = \frac{1}{\sqrt{1+|w|^2}} \begin{pmatrix} -w_1 w_i / 1+|w|^2 \\ \vdots \\ 1 - w_i^2 / 1+|w|^2 \\ \vdots \\ -w_n w_i / 1+|w|^2 \end{pmatrix} = \frac{1}{\sqrt{1+|w|^2}} \left[\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} - \frac{w_i}{1+|w|^2} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} \right]$$

$$= \frac{1}{\sqrt{1+|w|^2}}\left(e_i - \frac{w_i}{1+|w|^2} w\right)$$

With the representation (7) we see that, if $v \in \mathbb{R}^n$:

$$(M(w) v) \cdot v = \frac{1}{\sqrt{1+|w|^2}}\left(v - \frac{(w \cdot v) w}{1+|w|^2}\right) \cdot v = \frac{1}{\sqrt{1+|w|^2}}\left(|v|^2 - \frac{(v \cdot w)^2}{1+|w|^2}\right)$$

$$\geq \frac{1}{(1+|w|^2)^{1/2}}\left(|v|^2 - \frac{|v|^2 |w|^2}{1+|w|^2}\right),$$

by Schwartz inequality.

$\Rightarrow$

$$(M(\omega)v)\cdot v \geq \frac{|v|^2}{(1+|\omega|^2)^{3/2}}\left(1+\cancel{|\omega|^2}-\cancel{|\omega|^2}\right).$$

If $|\omega| \leq R \Rightarrow |\omega|^2 \leq R^2 \Rightarrow (1+|\omega|^2)^{3/2} \leq (1+R^2)^{3/2}$

$$\Rightarrow \frac{1}{(1+|\omega|^2)^{3/2}} \geq \frac{1}{(1+R^2)^{3/2}}$$

Hence:

$$\boxed{(M(\omega)v)\cdot v \geq \frac{|v|^2}{(1+R^2)^{3/2}} \quad \forall v \in \mathbb{R}^n,\ \forall \omega,\ |\omega| \leq R} \quad (8)$$

Using (8) one can prove the following:

<u>Theorem (Elliptic equations for directional derivatives)</u>:

Consider the area functional:

$$\mathcal{A}(u;B) = \int_B \sqrt{1+|\nabla u|^2}$$

Then:

(i) If u is a Lipschitz local minimizer of $\mathcal{A}$ in B, then u solves the weak Euler-Lagrange equation:

$$\int_B \frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\cdot \nabla \varphi = 0 \quad \forall \varphi \in C_c^\infty(B) \quad (9)$$

(ii) If u is a Lipschitz function solving (9), then $u \in W^{2,2}_{loc}(B)$, and for every $i = 1, \dots, n$, the distributional directional derivative $v = \partial_i u$ of u satisfies the elliptic equation in divergence form:

$$\boxed{\int_B A(x)\nabla v\cdot\nabla\varphi = 0 \quad \forall\varphi\in C_c^\infty(B) \qquad (10)}$$

where:
$$A(x) = D^2 f(\nabla u(x)).$$

Ideas for Proof:

From (9) we have that u satisfies:

$$-\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 \quad \text{on } B$$

Heuristically, if u is smooth, then we can differentiate this equation in the x_i direction, and commute div and ∂_i, to find:

$$-\operatorname{div}\left(D^2 f(\nabla u)\nabla(\partial_i u)\right) = 0 \quad \text{on } B$$

Let $v = \partial_i u$, $A(x) = D^2 f(\nabla u(x))$

and we have

$$-\operatorname{div}(A\nabla v) = 0 \quad \text{on } B$$

This can be made rigorous, for u being only Lipschitz, using the "diffence quotients method". Then, one can prove with this method that the validity of the Euler-Lagrange equation in weak form implies the existence of $v = \partial_i u$ as a distributional derivative in L^2 which solves the weak equation (10). For the details on this method, see the book "An introduction to the Regularity theory for Elliptic systems, Harmonic maps and minimal graphs", by M. Giaquinta and L. Martinazzi, Proposition 8.6, Page 172. $\blacksquare$

We now have the following:

<u>Theorem</u>: If E is a perimeter minimizer in the open set $A \subset \mathbb{R}^n$, then $A \cap \partial^* E$ is an analytic vanishing mean curvature hypersurface.

<u>Proof</u>: $\forall x \in A \cap \partial^* E$ and $\gamma \in (0,1)$ $\exists r > 0$ and $u: \mathbb{R}^{n-1} \to \mathbb{R}$, $\mathrm{Lip}(u) \le 1$ and $u \in C^{1,\gamma}(\underbrace{D(0,r)}_{= D_r})$ such that, <u>up to rotation</u>:

$$C(x,r) \cap \partial E = x + \{(z, u(z)) : z \in D_r\}$$

We have seen before that u is a minimizer of the area functional. By previous theorem, $u \in W^{2,2}(D_r)$ and $v = \partial_i u$ is a weak solution of:

$$(***)\begin{cases} -\operatorname{div}(A \nabla' v) = 0 \text{ in } D_r, \\ A = M_0(\nabla' u), \quad M(w) = D^2 f(w), \ w \in \mathbb{R}^{n-1}, \\ \quad f \text{ as in page } 35.8 \end{cases}$$

Since $u \in C^{1,\gamma}(D_r)$ and M is smooth $\Longrightarrow$

$$\boxed{A \in C^{0,\gamma}(D_r)}$$

By Schauder's theory we have:

$$\boxed{v \in C^{1,\gamma}(D_r)}$$

Since $v = \partial_i u$ and i is arbitrary $\Rightarrow \partial_1 u, \dots, \partial_n u$ are all in $C^{1,\gamma}(D_r)$.

$$\Longrightarrow \quad u \in C^{2,\gamma}(D_r)$$

and hence

$$A \in C^{1,\gamma}(D_r)$$

We apply Schauder's theory again to get $u \in C^{3,\gamma}(D_r)$.
We continue with this iteration to conclude:

$$u \in C^{\infty}(D_r).$$

Thus, u is a smooth solution of the minimal surface equation

$$-\operatorname{div}\left(\frac{\nabla' u}{\sqrt{1+|\nabla' u|^2}}\right)=0 \quad \text{on } D_r,$$

and thus u is analytic.

Another way to proceed in the first iteration is by using the theory of De Giorgi–Nash–Moser. Indeed, if we start with only:

$$u \text{ Lipschitz on } D_r$$

then:

$$A \in L^\infty(D_r)$$

$\Rightarrow$ by Giorgi–Nash–Moser that:

$$v \in C^{0,\alpha}(D_r)$$

$$\Rightarrow \qquad u \in C^{1,\alpha}(D_r),$$

and now we can continue as before to get:

$$u \in C^\infty(D_r.) \qquad \blacksquare$$