Lecture 36

Analysis of singularities

The following theorem is devoted to the study of the singular set:

$$\Sigma(E;A) = A \cap (\partial E \setminus \partial^* E)$$

aiming to provide estimates on its possible size:

<u>Theorem</u>: If E is a perimeter minimizer in the open set $A \subset \mathbb{R}^n$, $n \geq 2$, then:

(i) if $2 \leq n \leq 7$, then Σ is empty

(ii) If $n = 8$, then Σ has no accumulation points in A

(iii) If $n \geq 9$, then $\mathcal{H}^s(\Sigma) = 0 \ \forall s > n-8$.

Before going into details of this proof, we sketch the plan of the proof:

1. We start by looking at the blow-ups $E_{x,r}$, $x \in \Sigma$. Then $E_{x,r} \to K$ in L^1 and K is a cone with vertex at 0. This tangent cone K is not a half space an is singular at 0. Moreover, K minimizes perimeter in $\mathbb{R}^n$

2. We then prove that no singular minimizing cone exists in $\mathbb{R}^n$, if $n \leq 7$ (Simons' theorem).

3. We exhibit a singular minimizing cone with

a point vertex singularity in $\mathbb{R}^n$. This is known as the Simons cone.

4.- We introduce the so-called Federer's dimension reduction argument: If K is a singular minimizing cone in $\mathbb{R}^n$ and the singular set of K contains other points than its vertex, then $\exists K'$, a singular minimizing cone in $\mathbb{R}^{n-1}$

5.- Simons' theorem + Federer's argument imply (i) in theorem.

The proof of (i), (ii) and (iii) requires many intermediate results:

<u>Theorem</u> : <u>(Closure and local uniform convergence of singularities)</u>.

Let $\{E_k\}_{k=1}^{\infty}$, E perimeter minimizers in A with

$$E_k \to E \text{ in } L^1_{loc}, \quad x_k \in \Sigma(E_k; A), \quad x \in A \cap \partial E, \quad x_k \to x$$

then:

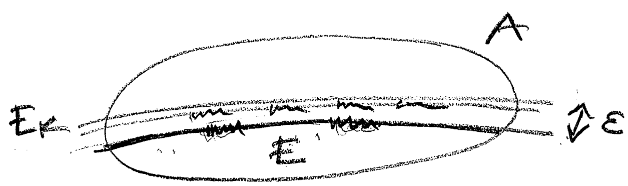
$$x \in \Sigma(E; A).$$

Moreover, given $\varepsilon > 0$ and $H \subset A$ compact, then $\exists k_0$ such that:

$$\Sigma(E_k; A) \cap H \subset I_\varepsilon(\Sigma(E; A) \cap H) \quad \forall k \geq k_0.$$

<u>Proof</u>: We must have $x \in \Sigma(E; A)$, for otherwise we would obtain a contradiction thanks to the theorem proved in previous lecture. For the second part of the theorem we proceed by contradiction. Then $\exists\, \varepsilon > 0$, $H \subset A$ compact, $k_j \to \infty$ as $j \to \infty$, $y_K \in \Sigma(E_{k_j}; A) \cap H$ such that:

$$d(y_K, \Sigma(E; A) \cap H) \geq \varepsilon.$$



$\{y_K\} \subset H \Rightarrow$ exists a subsequence, still labeled as $\{y_K\}$ such that:

$$y_K \to y, \quad y \in H \subset A.$$

By the first part of the theorem, $y \in \Sigma(E; A) \cap H$. In particular $\exists\, K_0$ s.t. $y_K \in B(y, \varepsilon) \subset I_\varepsilon(\Sigma(E; A) \cap H)$ for every $K \geq K_0$, which is a contradiction. ∎

Existence of densities at singular points.

At this point we recall the monotonicity of density ratios for perimeter minimizers. Indeed, if E minimizes perimeter in A, and $x \in \partial E \cap A$ then

$$\boxed{r \mapsto \frac{P(E; B(x,r))}{r^{n-1}} \text{ is increasing on } 0 < r < \operatorname{dist}(x, \partial A) \quad (*)}$$

We proved (*) using two methods. The first one, in Lecture 23 used the fact that E is stationary and hence $\int_{\partial^* E} \operatorname{div}_E T = 0$, $\forall T \in C_c^1(A)$.

The other proof, in Lecture 24, uses a comparison argument with a cone. With (*) at hand, we proved in Lecture 23 the existence of densities at any point $x \in \partial E \cap A$. Indeed, we showed in Lecture 23 that:

$$\lim_{r \to 0^+} \frac{P(E; B(x,r))}{\omega_{n-1} r^{n-1}} \text{ exists } \forall x \in \partial E \cap A. \text{ Moreover:}$$

$$\lim_{r \to 0^+} \frac{P(E; B(x,r))}{\omega_{n-1} r^{n-1}} \geq 1 \ \forall x \in \partial E \cap A \text{; and exactly equal}$$

to 1 if $x \in \partial^* E \cap A$.

Since $\Sigma(E; A) \subset A \cap \partial E$, we infer the existence of densities at any point $x \in \Sigma(E; A)$.

Blow-ups at singularities and tangential minimal cones.

<u>Def</u>: We say that $K \subset \mathbb{R}^n$ is a cone with vertex at $x \in \mathbb{R}^n$ if:

$$K = K_{x,r} = \frac{K - x}{r}, \quad \forall r > 0.$$

<u>Remark</u>: In the rest of the notes we always tacitly assume we are dealing with cones with vertex at 0.

<u>Def</u>: If a cone K is a perimeter minimizer in $\mathbb{R}^n$ and:

$$\Sigma(K) = \Sigma(K; \mathbb{R}^n) = \partial K \setminus \partial^* K \neq \emptyset$$

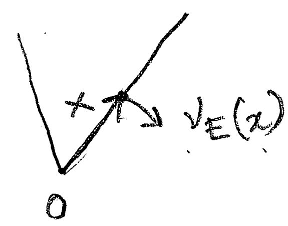
then K is a <u>singular minimizing cone</u>.

<u>Remark</u>: If K is a singular minimizing cone, then $0 \in \Sigma(K)$. Indeed, if $0 \notin \Sigma(K)$ then $K = K_{0,r} = \frac{K}{r}$ would locally converge to a half-space. That is, K would be a half-space, forcing $\Sigma(K) = \emptyset$, a contradiction.

We also need the following:

Proposition (Characterization of cones):
If E is a set of locally finite perimeter in $\mathbb{R}^n$, then $E^{(1)}$ is a cone if and only if, for $\mathcal{H}^{n-1}$-a.e. $x \in \partial^* E$,

$$x \cdot \nu_E(x) = 0 .$$



Another tool we will use is:

Theorem (Monotonicity formula): If E is stationary for the perimeter in A and $x_0 \in A \cap \partial E$, then for a.e. $r \in (0, \operatorname{dist}(x_0, \partial A))$,

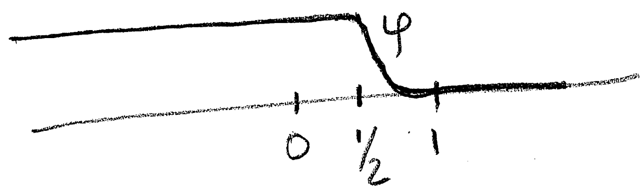
$$\frac{d}{dr} \frac{P(E; B(x_0,r))}{r^{n-1}} = \frac{d}{dr} \int_{B(x_0,r) \cap \partial^* E} \frac{(\nu_E(x) \cdot (x - x_0))^2}{|x - x_0|^{n+1}} \, d\mathcal{H}^{n-1}(x) .$$

Proof of the Monotonicity formula:
Define $d = \operatorname{dist}(x_0, \partial A)$.
WLOG we assume $x_0 = 0$, so that $B_r = B(x_0, r)$.
We proceed as in the proof of the monotonicity of density ratios proved in Lecture 23. We define:

$\varphi \in C^\infty(\mathbb{R}^n; [0,1])$, $\varphi \equiv 1$ on $(-\infty, \frac{1}{2})$

$\varphi \equiv 0$ on $(1, \infty)$

$\varphi' \leq 0$



Define:

$$\left.\begin{aligned} \Phi(r) &= \int_{\partial^* E} \varphi\left(\frac{|x|}{r}\right) d\mathcal{H}^{n-1}, \quad r \in (0,d) \\ \Psi(r) &= \int_{\partial^* E} \varphi\left(\frac{|x|}{r}\right) \frac{(x \cdot \nu_E(x))^2}{|x|^2} \, d\mathcal{H}^{n-1}(x) \end{aligned}\right\} \quad (1)$$

In Lecture 23, we proved:

$$\boxed{\begin{aligned} &(n-1)\int_{\partial^* E} \varphi\left(\frac{|x|}{r}\right) d\mathcal{H}^{n-1} + \int_{\partial^* E} \frac{|x|}{r} \varphi'\left(\frac{|x|}{r}\right) d\mathcal{H}^{n-1} \\ &\qquad = \int_{\partial^* E} \frac{|x|}{r} \varphi'\left(\frac{|x|}{r}\right) \frac{(x \cdot \nu_E(x))^2}{|x|^2} \, d\mathcal{H}^{n-1} \qquad (2) \end{aligned}}$$

From (1) and (2)

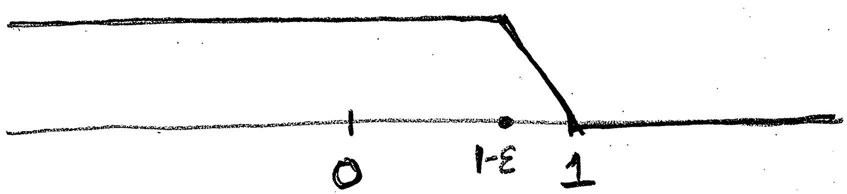
$$(n-1)\Phi(r) - r\Phi'(r) = -r\Psi'(r)$$

$$\Rightarrow \frac{r\Phi'(r)}{r^n} - \frac{(n-1)\Phi(r)}{r^n} = \frac{r\Psi'(r)}{r^n}$$

$$\Rightarrow \boxed{\frac{\Phi'(r)}{r^{n-1}} - \frac{(n-1)\Phi(r)}{r^n} = \frac{\Psi'(r)}{r^{n-1}} \quad \text{for a.e. } r \in (0,d)} \quad (3)$$

We now define the Lipschitz functions $\varphi_\varepsilon : \mathbb{R} \to [0,1]$, $\varepsilon \in (0,1)$, as:

$$\varphi_\varepsilon(s) = \chi_{(-\infty, 1-\varepsilon)}(s) + \frac{(1-s)}{\varepsilon} \chi_{(1-\varepsilon, 1)}(s), \quad s \in \mathbb{R}.$$



By an approximation argument, (3) is valid with φ_ε instead of φ. Denote Φ_ε and Ψ_ε the functions defined as in (1) with φ_ε:

$$\boxed{\frac{\Phi_\varepsilon'(r)}{r^{n-1}} - (n-1)\frac{\Phi_\varepsilon(r)}{r^n} = \frac{\Psi_\varepsilon'(r)}{r^{n-1}}, \text{ for a.e. } r \in (0,d)} \quad (4)$$

Define:

$$(5) \begin{cases} \Phi_0(r) = P(E; B_r) \\ \beta(r) = \displaystyle\int_{B_r \cap \partial^* E} \frac{(\nu_E(x) \cdot x)^2}{|x|^{n+1}} \, d\mathcal{H}^{n-1}(x), \quad r \in (0,d) \end{cases}$$

Claim: $\Phi_\varepsilon'(r) \to \Phi_0'(r)$, $\Psi_\varepsilon'(r) \to r^{n-1}\beta''(r)$ as $\varepsilon \to 0$, for a.e. $r \in (0,d)$ where both $\Phi_0(r)$ and $\Psi_\varepsilon(r)$ are differentiable.

With the claim we can finish the Monotonicity formula. Indeed, assume that the claim is true.

Then letting $\varepsilon \to 0$ in (4) yields;

$$\frac{\Phi_0'(r)}{r^{n-1}} - (n-1)\frac{\Phi_0(r)}{r^n} = \frac{r^{n-1}\cdot \beta'(r)}{r^{n-1}}$$

$$\Rightarrow \underbrace{\frac{\Phi_0'(r)}{r^{n-1}} - \frac{(n-1)\Phi_0(r)}{r^n}}_{\frac{d}{dr}\left(r^{1-n}\Phi_0(r)\right)} = \beta'(r)$$

That is;

$$\frac{d}{dr}\left(\frac{P(E;B_r)}{r^{n-1}}\right) = \frac{d}{dr}\int_{B_r\cap\partial^* E}\frac{(\nu_E(x)\cdot x)^2}{|x|^{n+1}}\,d\mathcal{H}^{n-1}(x)$$

for a.e. $r\in(0,d)$; which is the desired formula.

We only need to show the claim:

We compute;

$$\Phi_\varepsilon'(r) = \int_{\partial^* E}\frac{\partial}{\partial r}\left(\varphi_\varepsilon\left(\frac{|x|}{r}\right)\right)d\mathcal{H}^{n-1}(x)$$

$$= \int_{\partial^* E} -\frac{|x|}{r^2}\,\varphi_\varepsilon'\left(\frac{|x|}{r}\right)d\mathcal{H}^{n-1}(x)$$

$\frac{|x|}{r} < 1 \Rightarrow |x| < r$

$$= \int_{\partial^* E\cap[B_r\setminus B_{r(1-\varepsilon)}]} -\frac{|x|}{r^2}\cdot\left(-\frac{1}{\varepsilon}\right)d\mathcal{H}^{n-1}(x)$$

$\frac{|x|}{r} < 1-\varepsilon \Rightarrow |x| < (1-\varepsilon)r$

$$= \frac{1}{\varepsilon r}\int_{(B_r\setminus B_{r(1-\varepsilon)})\cap\partial^* E}\frac{|x|}{r}\,d\mathcal{H}^{n-1}(x) \qquad \forall r\in(0,d).$$

If $x\in B_r\setminus B_{r(1-\varepsilon)} \Rightarrow r(1-\varepsilon)\le |x|\le r$

$\Rightarrow 1-\varepsilon \le \frac{|x|}{r} \le 1$

$\Rightarrow$

$$\frac{(1-\varepsilon)}{\varepsilon r}\int_{(B_r\setminus B_{r(1-\varepsilon)})\cap\partial^* E} d\mathcal{H}^{n-1}(x) \leq \Phi_\varepsilon'(r) \leq \frac{1}{\varepsilon r}\int_{(B_r\setminus B_{r(1-\varepsilon)})\cap\partial^* E} d\mathcal{H}^{n-1}(x)$$

$\Rightarrow$

$$(1-\varepsilon)\frac{P(E;B_r)-P(E;B_{r-r\varepsilon})}{\varepsilon r} \leq \Phi_\varepsilon'(r) \leq \frac{P(E;B_r)-P(E;B_{r-r\varepsilon})}{\varepsilon r}$$

If Φ_0 is differentiable at r, letting $\varepsilon\to 0^+$ in previous inequality yields:

$$\Phi_0'(r) \leq \lim_{\varepsilon\to 0^+}\Phi_\varepsilon'(r) \leq \Phi_0'(r)$$

$$\Rightarrow \boxed{\lim_{\varepsilon\to 0^+}\Phi_\varepsilon'(r) = \Phi_0'(r)}$$

In the same way:

$$\frac{\Psi_\varepsilon'(r)}{r^{n-1}} = \frac{1}{r^{n-1}}\int_{\partial^* E} -\frac{|x|}{r^2}\varphi_\varepsilon'\left(\frac{|x|}{r}\right)\frac{(x\cdot\nu_E(x))^2}{|x|^2}\,d\mathcal{H}^{n-1}(x)$$

$$= \frac{1}{r^{n-1}}\int_{\partial^* E\cap(B_r\setminus B_{r(1-\varepsilon)})} -\frac{|x|}{r^2}\cdot\left(-\frac{1}{\varepsilon}\right)\frac{(x\cdot\nu_E(x))^2}{|x|^2}\,d\mathcal{H}^{n-1}(x)$$

$$= \frac{1}{\varepsilon r}\int_{\partial^* E\cap(B_r\setminus B_{r(1-\varepsilon)})} \frac{|x|}{r^n}\,\frac{(x\cdot\nu_E(x))^2}{|x|^2}\,d\mathcal{H}^{n-1}(x)$$

$$= \frac{1}{\varepsilon r}\int_{\partial^* E\cap(B_r\setminus B_{r(1-\varepsilon)})} \frac{|x|\,|x|^{n-1}}{r^n}\,\frac{(x\cdot\nu_E(x))^2}{|x|^2\,|x|^{n-1}}\,d\mathcal{H}^{n-1}(x)$$

$\Rightarrow$

$$\frac{\Psi_\varepsilon'(r)}{r^{n-1}} = \frac{1}{\varepsilon r}\int_{\partial^* E\cap(B_r\setminus B_{r(1-\varepsilon)})} \left(\frac{|x|}{r}\right)^n \frac{(x\cdot\nu_E(x))^2}{|x|^{n+1}}\, d\mathcal{H}^{n-1}(x).$$

Again:

$$x\in B_r\setminus B_{r(1-\varepsilon)} \Rightarrow r(1-\varepsilon)\le |x|\le r$$
$$\Rightarrow 1-\varepsilon \le \frac{|x|}{r}\le 1$$

$\Rightarrow$

$$(1-\varepsilon)^n\frac{1}{\varepsilon r}\int_{\partial^* E\cap(B_r\setminus B_{r(1-\varepsilon)})}\frac{(x\cdot\nu_E(x))^2}{|x|^{n+1}}\,d\mathcal{H}^{n-1}(x)\le \frac{\Psi_\varepsilon'(r)}{r^{n-1}}\le \frac{1}{\varepsilon r}\int_{\partial^* E\cap(B_r\setminus B_{r(1-\varepsilon)})}\frac{(x\cdot\nu_E(x))^2}{|x|^{n+1}}\,d\mathcal{H}^{n-1}(x)$$

$\Rightarrow$

$$(1-\varepsilon)^n\frac{\beta(r)-\beta(r-\varepsilon r)}{\varepsilon r}\le\frac{\Psi_\varepsilon'(r)}{r^{n-1}}\le\frac{\beta(r)-\beta(r-\varepsilon r)}{\varepsilon r}$$

If γ is differentiable at r, letting $\varepsilon\to 0^+$ in the previous inequality yields:

$$1\cdot\beta'(r)\le\lim_{\varepsilon\to 0^+}\frac{\Psi_\varepsilon'(r)}{r^{n-1}}\le\beta'(r)$$

$$\therefore\quad \lim_{\varepsilon\to 0^+} r^{1-n}\Psi_\varepsilon'(r) = \beta'(r)$$

or:

$$\Psi_\varepsilon'(r)\longrightarrow r^{n-1}\beta'(r),$$

which completes the proof of the claim. ∎

We can now prove:

<u>Theorem</u> (<u>Tangent singular minimizing cones</u>):

If E is a perimeter minimizer in an open set $A \subset \mathbb{R}^n$ and:

$$x \in \Sigma(E;A), \quad r_k \to 0, \quad E_k = E_{x,r_k} = \frac{E-x}{r_k}$$

then there exits a singular minimizing cone $K \subset \mathbb{R}^n$ an a subsequence $\{k_j\}$, $k_j \to \infty$ as $j \to \infty$ such that:

$$E_{k_j} \to K \text{ in } L^1_{loc} \quad \mu_{E_{k_j}} \xrightarrow{*} \mu_E, \quad |\mu_{E_{k_j}}| \xrightarrow{*} |\mu_K|.$$

<u>Proof</u>:

Fix $x \in \Sigma(E;A)$ and consider the sequence of blow-ups:

$$E_k = \frac{E-x}{r_k}, \quad A_k = \frac{A-x}{r_k}$$

We have explained before (see Lecture 28, Remark 1) that if E minimizes perimeter in A, then E_k minimizes perimeter in A_k; for each $k=1,2,\ldots$

Given $R>0$, $\exists\, k(R)$ such that:

$$B_R \subset\subset A_k, \quad \forall k \geq k(R).$$

$\{E_k\}_{k \geq k(R)}$ is a sequence of minimizers in B_R.

By the compactness for minimizers Lemma proved in Lecture 26 we have:

$\exists\, F_R \subset B_R$ and a subsequence $\{E_{k_j}\}$ s.t:

$E_{k_j} \cap B_R \to F_R$, $\mu_{E_{k_j}} \overset{*}{\rightharpoonup} \mu_{F_R}$, and $|\mu_{E_{k_j}}| \overset{*}{\rightharpoonup} |\mu_{F_R}|$, in B_R, F_R minimizes perimeter in B_R.

By a diagonal argument we obtain the existence of a set of locally finite perimeter F and a subsequence, that for simplicity we will denote again as E_{k_j} such that:

$$(5)\quad \begin{cases} E_{k_j} \to F \text{ in } L^1_{loc},\ \mu_{E_{k_j}} \overset{*}{\rightharpoonup} \mu_F,\ |\mu_{E_{k_j}}| \overset{*}{\rightharpoonup} |\mu_F|, \\ \text{as } j \to \infty, \text{ and } F \text{ is a perimeter minimizer} \\ \text{in } \mathbb{R}^n. \end{cases}$$

By the theorem proved at the beggining of this Lecture, since $0 \in \Sigma(E_k; A_k)$ then $0 \in \Sigma(F; \mathbb{R}^n)$.

<u>Claim</u>: F is a cone.

By weak convergence, for a.e. s such that $\mathcal{H}^{n-1}(\partial^* F \cap \partial B_s) = 0$ we have that:

$$\begin{aligned} P(F; B_s) &= \lim_{j \to \infty} P(E_{k_j}; B_s) \\ &= \lim_{j \to \infty} \frac{P(E; B(x, s r_{k_j}))}{r_{k_j}^{n-1}} \qquad (6) \end{aligned}$$

We remarked in Page 36.4 about existence of $(n-1)$-dimensional densities at every point $x \in \partial E$ that is a singular point. That is:

$$\Theta_{n-1}(\mu)(x) := \lim_{r \to 0} \frac{P(E; B(x,r))}{\omega_{n-1} r^{n-1}}, \quad \mu = \mathcal{H}^{n-1} \llcorner \partial^* E,$$

exists, for every $x \in \partial E$.

Therefore, from (6):

$$P(F; B_s) = \omega_{n-1} s^{n-1} \lim_{j \to \infty} \frac{P(E; B(x, s r_{k_j}))}{\omega_{n-1} (s r_{k_j})^{n-1}}$$

$$= \omega_{n-1} s^{n-1} \Theta_{n-1}(\mu)(x)$$

$$\Rightarrow \boxed{P(F; B_s) = \omega_{n-1} s^{n-1} \Theta_{n-1}(\mu)(x), \text{ for a.e. } s > 0} \quad (7)$$

From (7), we obtain that the $(n-1)$-dimensional density of $\partial^* F$ at $0 \in \partial F$ is constant. Indeed:

$$\frac{P(F; B_s)}{\omega_{n-1} s^{n-1}} = \Theta_{n-1}(\mathcal{H}^{n-1} \llcorner \partial^* F)(x) \quad \text{for a.e. } s > 0$$

Now, by the monotonicity formula:

$$0 = \int_{(B_s \setminus B_t) \cap \partial^* F} \frac{(\nu_F(y) \cdot y)^2}{|y|^{n+1}} \, d\mathcal{H}^{n-1}(y) = 0, \quad \text{for a.e. } 0 < s < t$$

Therefore, $\nu_F(y) \cdot y = 0$ for $\mathcal{H}^{n-1}$ a.e. $y \in \partial^* F$, and by the Proposition in Page 36.6 we conclude that F is a cone.