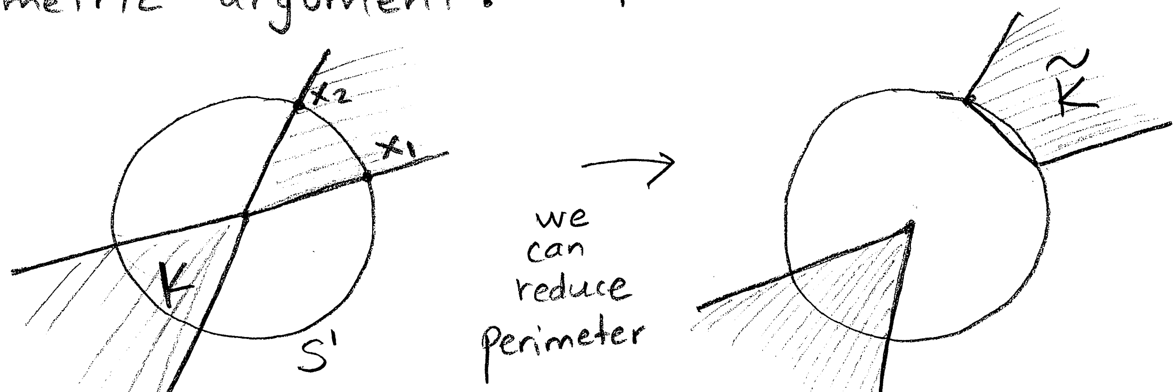
Lecture 37

Simon's theorem.

Theorem (Simons' theorem): If $n \geq 2$ and there exists a singular minimizing cone $K \subset \mathbb{R}^n$ with $\Sigma(K) = \{0\}$, then $n \geq 8$.

Main steps in the Proof of this theorem (for full details, consult the book by E. Giusti on "Minimal surfaces and functions of bounded variation" or the book by L. Simon: "Lectures on geometric measure theory".

Note that for $n=2$, we can give a simple geometric argument. Suppose K is minimal:

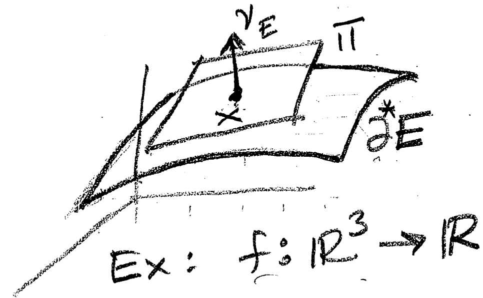


The proof in general is based on choosing appropiate test functions in the second variation of the perimeter, which we now discuss.

Recall, if E is a set of locally finite perimeter, $f \in C^1(\mathbb{R}^n; \mathbb{R}^m)$ then:

$\nabla^{\partial^* E} f$ is a linear map which is the restriction of the linear map ∇f to the tangent space $\Pi = \nu_E^{\perp}$; that is,

$$\nabla^{\partial^* E} f = \nabla f \llcorner \Pi .$$



Ex: $f: \mathbb{R}^3 \to \mathbb{R}$

In Lecture 24, we presented some examples to better understand $\nabla^{\partial^* E} f$. In particular, we showed that, if $\{\tau_1(x), \ldots, \tau_{n-1}(x)\}$ is an orthormal basis of $\nu_E(x)^{\perp}$ then:

$$\nabla f(x) = \sum_{i=1}^{n-1} \left(\nabla f(x) \tau_i(x)\right) \otimes \tau_i(x) + \left(\nabla f(x) \nu_E(x)\right) \otimes \nu_E(x)$$

(Recall $(w \otimes v) x = (v \cdot x) w$)

And:

$\nabla^{\partial^* E} f(x) = \nabla f(x) - \left(\nabla f(x) \nu_E(x)\right) \otimes \nu_E(x)$, as linear maps. If we now consider $\nabla f(x)$ as a vector, we have:

$$\boxed{\nabla^{\partial^* E} f(x) = \nabla f(x) - \left(\nabla f(x) \cdot \nu_E(x)\right) \nu_E(x)}$$

Let E be an open set with C^2-boundary in A. Then $\exists A'$ open with:

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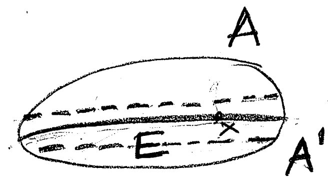
$$A \cap \partial E \subset A' \subset A,$$

such that the signed distance function:

$$s_E = \begin{cases} \operatorname{dist}(x, \partial E), & x \in \mathbb{R}^n \setminus E \\ -\operatorname{dist}(x, \partial E), & x \in E \end{cases}$$

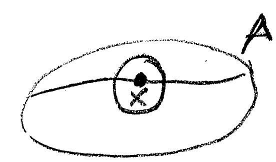


satisfies $s_E \in C^2(A')$.

Define now:

$$(1) \begin{cases} N_E = \nabla s_E, & \text{note that } N_E \in C^1(A'; \mathbb{R}^n) \\ A_E = D^2 s_E = \nabla(N_E), & \text{on } A' \end{cases}$$

• N_E is an extension to A' of the outer unit normal ν_E to E, with $|N_E| \equiv 1$ on A'.



• $\forall x \in A \cap \partial E$, $\exists r(x)$, $\exists \{\tau_i\}_{i=1}^{n-1} \subset C^1(B(x,r); S^{n-1})$

$\exists \{k_i\}_{i=1}^{n-1} \subset C(B(x,r))$

such that

(a) $\{\tau_1(y), \dots, \tau_{n-1}(y)\}$ is an orthonormal basis of $\nu_E(x)^{\perp} = T_x \partial E$ $\forall y \in B(x,r) \cap \partial E$

(b) $\{\tau_1(y), \dots, \tau_{n-1}(y)\} \cup \{N_E(y)\}$ is an orthonormal basis of $\mathbb{R}^n$ for every $y \in B(x,r)$.

(c) $A_E(y) = \sum_{i=1}^{n-1} k_i(y) \, \tau_i(y) \otimes \tau_i(y)$, $\forall y \in B(x,r)$

We define now

$$\boxed{H_E(y) = \sum_{i=1}^{n-1} k_i(y) = tr\,(A_E(y)), \ \forall y \in B(x,r) \cap \partial E}$$

<u>Remark</u>: The numbers $k_1(y), \dots, k_{n-1}(y)$ are the principal curvatures of ∂E at y, and $H_E(y)$ is the scalar mean curvature.

If $y \in B(x,r) \cap \partial E$, then $A_E(y)$ is a symmetric linear map:

$$A_E(y) : T_y \partial E \to T_y \partial E,$$

and is called the second fundamental form of ∂E at y.

Recall that we are considering E an open set with C^2- boundary in the open set A. Now, if $\xi \in C_c^\infty(A)$ and $\{f_t\}_{|t|<\varepsilon}$ is a local variation associated with the normal vector field $T := \xi N_E \in C_c^1(A; \mathbb{R}^n)$; then it can be proven that:

$$\boxed{\left.\frac{d^2}{dt^2}\right|_{t=0} P(f_t(E); A) = \int_{\partial E} |\nabla^{\partial E} \xi|^2 + (H_E^2 - |A_E|^2)\xi^2 \, d\mathcal{H}^{n-1}}$$

<u>Second variation of perimeter</u>,

where the norm of the linear map $A_E(y)$ is defined as follows;

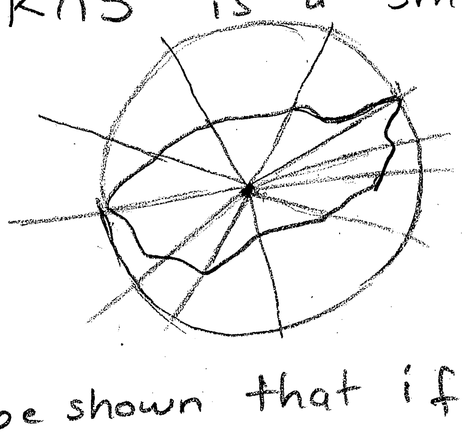
$$|A_E(y)| = \sqrt{\text{trace}\,(A_E(y)^* A_E(y))}$$

$$= \sqrt{\text{trace}\,(A_E(y)^2)} \quad ; \quad \text{since } A_E(y)^* = A_E(y), \text{ because } A_E(y) \text{ is symmetric}$$

$$\therefore \boxed{\begin{aligned} |A_E(y)|^2 &= \text{trace}\,(A_E(y))^2 \\ &= \sum_{i=1}^{n-1} \kappa_i(y)^2, \quad \forall y \in B(x,r)\cap\partial E \end{aligned}}$$

We now go back to the proof of Simons' theorem. The case $n=3$ can also be treated with a geometrical argument. Indeed, let K be a minimizing cone in $\mathbb{R}^3$ with $\Sigma(K)=\{0\}$.

$\Rightarrow$ $\partial K \cap S^2$ is a smooth curve



It can be shown that if E is an open set with C^2-boundary then:

$$(***) \quad \int_{\partial E} \text{div}_E T \, d\mathcal{H}^{n-1} = \int_{\partial E} (T\cdot\nu_E) H_E \, d\mathcal{H}^{n-1} \quad \forall T \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$$

We recall that if E minimizes perimeter in A then

$$\int_{\partial E} \mathrm{div}_E T \cdot d\mathcal{H}^{n-1} = 0 \quad \forall T \in C_c^1(A; \mathbb{R}^n).$$

Thus, if E is a perimeter minimizer in A, then:

$$\boxed{H_E(y) = 0 \quad \forall y \in A \cap \partial E} \qquad (2)$$

Apply (2) to the minimal cone $K \subset \mathbb{R}^3$ to get:

$$H_K(y) = 0 \implies k_1(y) + k_2(y) = 0 \quad \forall y \in \partial K \setminus \{0\}.$$

Now, $\forall y \in \partial K \setminus \{0\}$, a line passes through x and lies in K. So $k_1(y) = 0$ or $k_2(y) = 0$. But if one is zero, then the other must be zero too.

$$\implies k_1(y) = k_2(y) = 0 \quad \forall y \in \partial K \setminus \{0\}$$

$\implies \partial K$ is a plane $\implies \Sigma(K) = \emptyset \implies$ contradiction.

We have then shown that, if $n=3$, there are <u>not</u> singular minimizing cones $K \subset \mathbb{R}^n$ with $\Sigma(K) = \{0\}$, which is Simons' theorem for $n=3$.

<u>Remark</u>: K has a singularity at but we can still apply (***) in previous page if we smooth out K around 0 in smaller and smaller neighborhoods.

For the general case, $3 \leq n \leq 7$, let $K \subset \mathbb{R}^n$ be a singular minimizing cone with

$$\Sigma(K) = \{0\}.$$

(**)

Since K minimizes perimeter in $\mathbb{R}^n$ then $H_K(x) = 0$, $\forall x \in \partial K \setminus \{0\}$. Moreover, $\frac{d^2}{dt^2}\Big|_{t=0} P(f_t(E); A) = 0$ along $T = \zeta N_E$. Then, the second variation of perimeter yields:

$$(3) \quad \boxed{\int_{\partial K} |\nabla^{\partial K} \zeta|^2 \, d\mathcal{H}^{n-1} \geq \int_{\partial K} |A_K|^2 \zeta^2 \, d\mathcal{H}^{n-1}}, \quad \forall \zeta \in C_c^1(\mathbb{R}^n), \ \operatorname{spt}\zeta \cap \{0\} = \emptyset$$

We want to deduce from (3) that $|A_K(x)| = 0$, $\forall x \in \partial K \setminus \{0\}$.

Choose:

$$\boxed{\zeta(x) = \varphi(x)\, |A_K(x)|, \quad \varphi \in C_c^\infty(\mathbb{R}^n), \ \operatorname{spt}\varphi \cap \{0\} = \emptyset} \quad (4)$$

In order to settle possible smoothness issues, we replace $|A_K|$ with $\sqrt{|A_K|^2 + \varepsilon}$ and then let $\varepsilon \to 0^+$ at the end of the argument.

Plugging (4) in (3) yields (after a long computation):

$$\boxed{(5) \quad \int_{\partial K} |A_K|^2 \left(|\nabla^{\partial K} \varphi|^2 - 2\frac{\varphi^2}{|x|^2} \right) d\mathcal{H}^{n-1} \geq 0, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^n), \ \operatorname{spt}\varphi \cap \{0\} = \emptyset}$$

Since:

$x \mapsto \nu_K(x)$ is 1-homogeneous.

$\Rightarrow x \mapsto \underset{\nabla \nu_K(x)}{\overset{\shortparallel}{A_K(x)}}$ is (-1)-homogeneous.

$$\Rightarrow \quad |A_K(x)| \leq \frac{c}{|x|}, \quad \forall x \in \partial K \setminus \{0\}$$

$\Rightarrow$

$$(6) \quad \int_{\partial K} |\nabla^{\partial K} \varphi|^2 |A_K|^2 \geq 2 \int \frac{\varphi^2}{|x|^2} |A_K|^2, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^n), \ \operatorname{spt} \varphi \cap \{0\} = \emptyset$$

The previous inequality is true, by approximation, for any $\varphi : \mathbb{R}^n \to \mathbb{R}$ Lipschitz satisfying:

$$\int_{\partial K} \frac{\varphi(x)^2}{|x|^4} \, d\mathcal{H}^{n-1}(x) < \infty \ ;$$

Choose $\varphi = u(|x|)$ radial function.

Since $x \cdot \nu_K(x) = 0$, notice that:

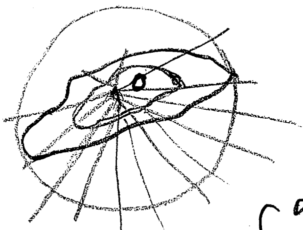
$$\nabla^{\partial K}(|x|) = \nabla(|x|) - \left(\nabla(|x|) \cdot \nu_K(x)\right) \nu_K(x)$$
$$= \frac{x}{|x|} - \left(\frac{x}{|x|} \cdot \nu_K(x)\right) \nu_K(x) = \frac{x}{|x|} \Rightarrow |\nabla^{\partial K} f(x)| = 1, \quad f(x) = |x|.$$

Recall the coarea formula for rectifiable sets:

$$\int_{\partial K} g \, |\nabla^{\partial K} f| \, d\mathcal{H}^{n-1} = \int_{\mathbb{R}} \left(\int_{\partial K \cap \{f = t\}} g \, d\mathcal{H}^{n-2} \right), \quad f \text{ Lipschitz.}$$

$\Rightarrow$ With $g = u(|x|)^2/|x|^4$ and $f = |x|$ in previous formula:

$$\int_{\partial K} \frac{\varphi^2}{|x|^4} \, d\mathcal{H}^{n-1} = \int_0^\infty \left(\int_{\partial K \cap \partial B_r} \frac{u(r)^2}{r^4} \, d\mathcal{H}^{n-2} \right) dr \qquad (37.9)$$



$$= \int_0^\infty \left(\int_{\partial K \cap \partial B_1} \frac{u(r)^2}{r^4} \, d\mathcal{H}^{n-2} \right) r^{n-2} \, dr$$

Coarea factor, Recall that ∂K is $(n-1)$-dimensional.

$$= \mathcal{H}^{n-2}(\partial K \cap S^{n-1}) \int_0^\infty u(r)^2 r^{n-6} \, dr.$$

<u>Remark</u>: In the previous equality, we have used that r^{n-2} is the coarea factor to transfer the integral to ∂B_1. This is because ∂K is $(n-1)$-dimensional. For n-dimensional regions of integration, the factor is r^{n-1}. Indeed:

$$\int_{\mathbb{R}^n} g \, dx = \int_0^\infty \left(\int_{\partial B_r} g(y) \, d\mathcal{H}^{n-1}(y) \right) dr$$

$$= \int_0^\infty \left(\int_{\partial B_1} g(y) \, d\mathcal{H}^{n-1}(y) \right) r^{n-1} \, dr.$$

For $\varphi(x) = u(|x|)$, $\nabla^{\partial K} \varphi(x) = \nabla \varphi(x) - (\nabla \varphi(x) \cdot \nu_K(x)) \nu_K(x)$.

Since $\nabla \varphi(x) = u'(|x|) \frac{x}{|x|}$ then, as before, since:

$$u'(x) \frac{x}{|x|} \cdot \nu_K(x) = 0 \qquad (K \text{ is a cone})$$

we have $|\nabla^{\partial K} \varphi(x)| = |\nabla \varphi(x)| = |u'(|x|)|$

Then, from (6):

$$(*)\begin{cases}\displaystyle\int_{\partial K}|A_K|^2\left(|u'(|x|)|^2-2\,\frac{u(|x|)^2}{|x|^2}\right)d\mathcal{H}^{n-1}(x)\geq 0,\\[2ex] \text{whenever } \displaystyle\int_0^\infty u(r)^2\,r^{n-6}\,dr<\infty\end{cases}$$

Consider $u:(0,\infty)\to\mathbb{R}$ as:

$$u(r)=\begin{cases}r^\alpha, & 0<r<1\\ r^\beta, & r>1\end{cases},\qquad \alpha,\beta\in\mathbb{R}.$$

Since $\int_0^\infty u(r)^2 r^{n-6}\,dr$ must be finite then we impose:

$$\int_0^1 r^{2\alpha}\cdot r^{n-6}\,dr+\int_1^\infty r^{2\beta}r^{n-6}\,dr<\infty$$

$$\Rightarrow \int_0^1 r^{2\alpha+n-6}+\int_1^\infty r^{2\beta+n-6}\,dr<\infty$$

$$\Rightarrow \quad 2\alpha+n-6>-1 \quad\text{and}\quad 2\beta+n-6<-1$$

$$\Rightarrow \quad 2\alpha+n>5 \quad\text{and}\quad 2\beta+n<5$$

$$\Rightarrow \quad \alpha>\frac{5-n}{2} \quad\text{and}\quad \beta<\frac{5-n}{2}$$

$$\Rightarrow \quad \boxed{\beta<\frac{5-n}{2}<\alpha} \quad (7)$$

Plugging $u(r)$ in $(*)$ above:

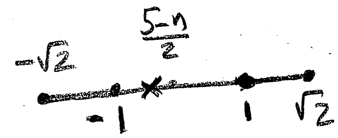
$$\boxed{0\leq(\alpha^2-2)\int_{B_1\cap\partial K}|A_K|^2|x|^{2(\alpha-1)}d\mathcal{H}^{n-1}+(\beta^2-2)\int_{\partial K\setminus B_1}|A_K|^2|x|^{2(\beta-1)}d\mathcal{H}^{n-1} \quad \text{if } \beta<\frac{5-n}{2}<\alpha.} \quad (8)$$

We are considering Simons' theorem for the case:

$$3 \le n \le 7.$$

We can choose α, β s.t. $\beta < \frac{5-n}{2} < \alpha$ and $\alpha^2 < 2$, $\beta^2 < 2$.

$$3 \le n \le 7 \implies -7 \le -n \le -3$$
$$-2 \le 5-n \le 2$$
$$-1 \le \frac{5-n}{2} \le 1$$



$$\alpha^2, \beta^2 < 2 \implies |\alpha|, |\beta| < \sqrt{2}$$

Choose $\alpha = \frac{6}{5}$, $\beta = -\frac{6}{5}$

$$\implies \alpha^2 < 2, \quad \beta^2 < 2, \quad \beta < -1 \le \frac{5-n}{2} \le 1 < \alpha$$

With this choice of α, β we have from (8) that

$$0 \le \underbrace{\left(\frac{36}{25} - 2\right)}_{<0} \int_{B_1 \cap \partial K} |A_K|^2 |x|^{2(\alpha-1)} \, d\mathcal{H}^{n-1} + \underbrace{\left(\frac{36}{25} - 2\right)}_{<0} \int_{\partial K \setminus B_1} |A_K|^2 |x|^{2(\beta-1)} \, d\mathcal{H}^{n-1}$$

Therefore, we must have

$$\int_{B_1 \cap \partial K} |A_K|^2 |x|^{2(\alpha-1)} \, d\mathcal{H}^{n-1}(x) = 0 = \int_{\partial K \setminus B_1} |A_K|^2 |x|^{2(\beta-1)} \, d\mathcal{H}^{n-1}(x).$$

Hence:

$|A_K(x)|^2 = 0, \quad \forall x \in \partial K \setminus \{0\}$

Recall that:

- $|A_K(x)|^2 = \sum_{i=1}^{n-1} k_i(x)^2$, where $k_i(x)$ are the principal curvatures at x
- $A_K(x) = \nabla \nu_K(x)$

Hence $|A_K(x)|^2 = 0$ implies that all the principal curvatures are 0. That is, $k_i(x) = 0 \ \forall i \in \{1, \dots, n-1\}, \forall x \in \partial K \setminus \{0\}$.

$\Rightarrow$ K is a half-space

$\Rightarrow \Sigma(K) = \emptyset$,

But this contradicts (**). We conclude that also for $3 \le n \le 7$, there doesn't exist singular minimimizing cones with $\Sigma(K) = \{0\}$

We saw at the beginning of this lecture, that this is also true for $n=2$ and $n=3$. Indeed, for this particular cases, a simple geometric argument was enough.

We conclude that for $2 \le n \le 7$, we can not find any singular minimizing cone in $\mathbb{R}^n$ with only one singularity (the vertex). ∎