Lecture 38

Federer's dimension reduction argument.

Simons' theorem excludes the existence in $\mathbb{R}^n$ of singular minimizing cones having just one singular point (the vertex), provided $2 \le n \le 7$. We now need to investigate the structure of singular minimizing cones with possibly larger singular sets.

The main idea is to blow-up such sets around their non-vertex singularities to prove the existence of lower dimensional singular minimizing cones.

<u>Theorem (Federer's dimension reduction theorem)</u>:

Let K be a singular minimizing cone in $\mathbb{R}^n$, $x_0 \in \Sigma(K)$, $x_0 \neq 0$. Let $r_k \to 0$. Consider the sequence of blow-ups $K_{x_0, r_k} = \dfrac{K - x_0}{r_k}$.

Then, up to extracting a subsequence and up to rotation, the blow-ups K_{x_0,r_k} locally converge to a cylinder $F \times \mathbb{R}$, where F is a singular minimizing cone in $\mathbb{R}^{n-1}$.

The proof of Federe's theorem requires the following two lemmas:

Lemma 1 (Half-lines of singular points). If K is a singular minimizing cone in $\mathbb{R}^n$, $x_0 \in \Sigma(K)$, $x_0 \neq 0$. Then:

$$\{\lambda x_0 : \lambda > 0\} \subset \Sigma(K) \text{ and } n \geq 3.$$

Lemma 2: (Cylinders of locally finite perimeters):

(i) If F is of locally finite perimeter in $\mathbb{R}^{n-1}$, then $F \times \mathbb{R}$ is of locally finite perimeter in $\mathbb{R}^n$, with:

$$\mu_{F \times \mathbb{R}} = (\nu_F(px), 0)\,\mathcal{H}^{n-1} \llcorner ((\partial^* F) \times \mathbb{R})$$

Moreover, if F is a perimeter minimizer in $\mathbb{R}^{n-1}$, then $F \times \mathbb{R}$ is a perimeter minimizer in $\mathbb{R}^n$.

(ii) If E is a set of locally finite perimeter in $\mathbb{R}^n$ such that:

$$\nu_E(x) \cdot e_n = 0 \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial^* E,$$

then $\exists F$, a set of locally finite perimeter in $\mathbb{R}^{n-1}$ such that $|E \Delta (F \times \mathbb{R})| = 0$.

If, moreover, E is a perimeter minimizer in $\mathbb{R}^n$, then F is a perimeter minimizer in $\mathbb{R}^{n-1}$.

With Simons' theorem and Federer's theorem, we can conclude the proof of the analysis of singularities, part (i).

Proof of the main theorem on Analysis of singularities, Part (i):

Let E be a perimeter minimizer in A, with:

$$\boxed{2 \leq n \leq 7} \quad (*)$$

If $\exists x \in \Sigma(E;A)$, then we proved in Lecture 36 that, blowing-up around x and passing to the limits yields the existence of a singular minimizing cone K in $\mathbb{R}^n$.

By Simons' theorem:

$$\boxed{\mathcal{H}^0(\Sigma(K)) > 1} \quad (1)$$

From (1) and Lemma 1 we must have $\boxed{n \geq 3}$.

We now perform a second blow-up around a point $y \in \Sigma(K)$, $y \neq 0$. After passing to the limit we obtain the existence of a singular minimizing cone K_1 in $\mathbb{R}^{n-1}$ (by Federer's dimension reduction argument). Applying Simon's theorem again yields:

$$\boxed{\mathcal{H}^0(\Sigma(K_1)) > 1} \quad (2)$$

From (2) and Lemma 1 we must have:

$$n-1 \geq 3,$$

which implies $n \geq 4$. By repeating this argument four more times, we find that:

$n \geq 8$, which contradicts (*). We conclude that:

$$\Sigma(E;A) = \emptyset. \quad \blacksquare$$

Proof of the main theorem on Analysis of Singularities, Part (ii):

Let $E \subset \mathbb{R}^8$ be a perimeter minimizer in some open set A.

Assume by contradiction that:

$\exists \{x_n\}$, $x_n \in \Sigma(E;A)$ such that $x_n \to x$, for some $x \in A \cap \partial E$.

We proved in Lecture 36 that $x_n \to x$ implies that x is also a singular point; that is $x \in \Sigma(E;A)$.

Let:

$$r_n = |x_n - x|,$$

and consider the sequence of blow-ups:

$$\{E_{x, r_n}\}$$

Then, up to possible extracting a subsequence, the blow-ups $\{E_{x,r_n}\}$ locally converges to a singular minimizing cone K in $\mathbb{R}^n$.

Up to extracting a further subsequence, we can assume:

$$\frac{x_n - x}{r_n} = \frac{x_n - x}{|x_n - x|} \longrightarrow z, \text{ for some } |z| = 1.$$

Since $\dfrac{x_n - x}{r_n} \in \Sigma(E_{x,r_n}; A_n)$, then again by the theorem proved in Lecture 36, Page 36.2, we must have that:

$$z \in \Sigma(K); \quad \text{recall } E_{x,r_n} \xrightarrow{loc} K$$

Since $z \neq 0$, because $|z| = 1$, then

$$\{0, z\} \subseteq \Sigma(K),$$

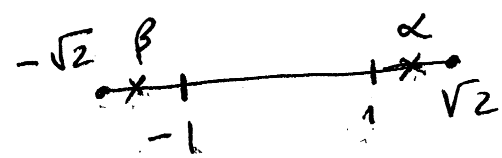
and by Federer's theorem (by blowing-up around z), we obtain the existence of a singular minimizing cone in $\mathbb{R}^7$. But this contradicts our main theorem part (i) on our main theorem of analysis of singularities. ∎

We have just shown that, for $n=8$, $\Sigma(E;A)$ has no accumulation points. We can now ask the question, whether there exists a singular minimizing cone $K \subset \mathbb{R}^8$ such that $\Sigma(K)=\{0\}$. In Simons' theorem we proved that such cone does not exist for $2 \leq n \leq 7$. Indeed, if we recall the proof of Simons' theorem in previous lecture we had:

$$3 \leq n \leq 7 \quad \Rightarrow -1 \leq \frac{5-n}{2} \leq 1$$

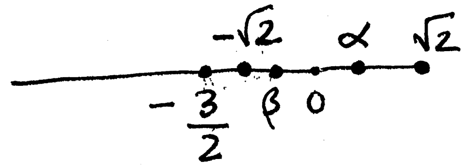
and we needed α, β, $\alpha^2 < 2$, $\beta^2 < 2$ such that

$$\beta < \frac{5-n}{2} < \alpha$$



So we had room to choose α, β. But just on the critical $n=8$, we have:

$$\frac{5-n}{2} = \frac{5-8}{2} = -\frac{3}{2}$$



So if α, β are such that $\alpha^2 < 2$, we can not meet the requirement $\beta < -3/2$; and the proof breaks. But then, actually one can construct a singular minimizing cone in $\mathbb{R}^8$ with $\Sigma(K)=\{0\}$!

Indeed, we have:

<u>Theorem</u>: (Examples of singular minimizing cones).
If $m \geq 4$, then the Simons cone

$$K = \{(x,y) \in \mathbb{R}^m \times \mathbb{R}^m : |x| < |y|\}$$

is a singular minimizing cone in $\mathbb{R}^m \times \mathbb{R}^m$.

This theorem can be proven using the <u>Calibration method</u>. See our textbook for more details.

Consider $m=4$. Then, by the above theorem:

$$K = \{(x,y) \in \mathbb{R}^4 \times \mathbb{R}^4 = \mathbb{R}^8 : |x| < |y|\}$$

is a singular minimizing cone in $\mathbb{R}^8$.
Moreover, note that $\Sigma(K) = \{0\}$; that is, $\mathcal{H}^0(\Sigma(K)) = 1$. Indeed, if $\mathcal{H}^0(\Sigma(K)) > 1$, then by Federer's theorem we get the existence of a singular minimizing cone $\tilde{K}$ in $\mathbb{R}^7$; and hence $\Sigma(\tilde{K}) \neq \emptyset$. But, on the other hand, our main theorem part (i) implies that $\Sigma(\tilde{K}) = \emptyset$ because $\tilde{K}$ minimizes perimeter in $\mathbb{R}^7$, and we have a contradiction.
We conclude then that $K = \{(x,y) \in \mathbb{R}^4 \times \mathbb{R}^4 : |x| < |y|\}$ is a singular minimizing cone in $\mathbb{R}^8$ with only one singularity (the vertex), which show that part (ii) of our main theorem is sharp.

Proof of the main theorem. Analysis of singularities, Part (iii):

The Haussdorf estimates:

$$\boxed{\mathcal{H}^s(\Sigma(E;A))=0 \quad \forall s>n-8,\ n\geq 9} \quad (*)$$

is based on covering arguments (see textbook for details).

Note that (*) implies that:

$$\boxed{\dim(\Sigma(E;A))\leq n-8, \quad n\geq 9} \quad (**)$$

We now proceed to show that the dimensional estimates (*) are sharp. Indeed, fix $n\geq 9$. Consider the set:

$$\tilde{E}=K\times\mathbb{R}^{n-8}, \text{ where } K=\{(x,y)\in\mathbb{R}^4\times\mathbb{R}^4 : |x|<|y|\}.$$

By Lemma 2, part (i), it follows that $\tilde{E}$ is a perimeter minimizer in $\mathbb{R}^n$ $(8+(n-8)=n)$.

Take $(0,x)\in\{0\}\times\mathbb{R}^{n-8}$ and consider the blow-ups:

$$\tilde{E}_{x,r}=\frac{\tilde{E}-x}{r}=\tilde{E}, \quad r>0$$

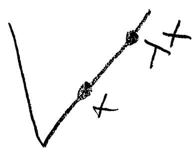
Clearly, $(0,0)\in\Sigma(\tilde{E})$. Hence $(0,0)\in\Sigma(\tilde{E}_{x,r})\Rightarrow$ $(0,x)\in\Sigma(\tilde{E})$. Since x was arbitrary we conclude that $\{0\}\times\mathbb{R}^{n-8}\subset\Sigma(\tilde{E})$. Hence, $\mathcal{H}^{n-8}(\Sigma(\tilde{E}))=\infty$, which says that $\dim(\tilde{E})=n-8$. Hence $\tilde{E}\subset\mathbb{R}^n$, $n>9$, minimizes perimeter in $\mathbb{R}^n$ and $\dim(\tilde{E})=n-8$, which shows that (**) is sharp. ∎

Proof of Federer's dimension reduction argument:

Proof of Lemma 1 (Half-lines of singular points):

<u>Step one</u>: We show that, if $x \in \partial^* K$ and $\lambda > 0 \Rightarrow$

$$\boxed{\nu_K(\lambda x) = \nu_K(x)}.$$



Let $\varphi \in C_c^1(\mathbb{R}^n)$, $\varphi_\lambda(y) = \varphi\left(\frac{y}{\lambda}\right)$, $y \in \mathbb{R}^n$, then:

$$\int_{\partial^* K} \varphi \nu_K \, d\mathcal{H}^{n-1} = \int_{\mathbb{R}^n} \varphi \, d\mu_K = \int_K \nabla\varphi = \int_{\lambda K} \nabla \varphi = \lambda^{1-n} \int_K \nabla \varphi_\lambda = \lambda^{1-n} \int_{\mathbb{R}^n} \varphi_\lambda \, d\mu_K$$

$$\| \quad \lambda^{1-n} \int_{\partial^* K} \varphi_\lambda \nu_K \, d\mathcal{H}^{n-1}$$

By approximation:

$$\therefore \quad \frac{1}{\omega_{n-1} r^{n-1}} \int_{B(x,r) \cap \partial^* K} \nu_K \, d\mathcal{H}^{n-1} = \frac{1}{\omega_{n-1} (\lambda r)^{n-1}} \int_{B(\lambda x, \lambda r) \cap \partial^* K} \nu_K \, d\mathcal{H}^{n-1}$$

Letting $r \to 0^+$:

$$\nu_K(x) = \nu_K(\lambda x)$$

<u>Step two</u>: Step one plus a change of variables $\Rightarrow$:

$$\frac{1}{r^{n-1}} \int_{B(x_0,r) \cap \partial^* K} \frac{|\nu_K - \nu|^2}{2} \, d\mathcal{H}^{n-1} = \frac{1}{(\lambda r)^{n-1}} \int_{B(\lambda x_0, \lambda r) \cap \partial^* K} \frac{|\nu_K - \nu|^2}{2} \, d\mathcal{H}^{n-1}, \quad \forall \nu$$

If $x_0 \in \Sigma(K)$, then by the characterization of singular sets proved in a previous lecture:

$$e(K, x_0, r) \geq \varepsilon(n), \quad \forall r>0$$

But: $e(K, x_0, r) = e(K, \lambda x_0, \lambda r)$; we proved it in previous page.

$$\Rightarrow \quad e(K, \lambda x_0, \lambda r) \geq \varepsilon(n), \ \forall r>0$$

$$\Rightarrow \quad \lambda x_0 \in \Sigma(K)$$

$$\therefore \quad \boxed{x_0 \in \Sigma(K) \Rightarrow \{\lambda x_0 : \lambda>0\} \subset \Sigma(K)}$$

In particular, $\boxed{\mathcal{H}^1(\Sigma(K)) = \infty}$ (3)

But recall that we have proved before that if E minimizes perimeter in A then $\mathcal{H}^{n-1}(A \cap (\partial E \setminus \partial^* E)) = 0$.

Hence, our singular minimizing cone K in $\mathbb{R}^n$ must satisfy $\mathcal{H}^{n-1}(\Sigma(K)) = 0$, which in view of (3) implies that $n-1>1 \Rightarrow n>2 \Rightarrow n \geq 3$. ∎

Proof of the Federer's dimension reduction argument (continuation):

K is a minimal cone in $\mathbb{R}^n$ with $\mathcal{H}^0(\Sigma(K)) > 1$.

Then $\exists x_0 \in \Sigma(K)$, $x_0 \neq 0$.

Consider the sequence of blow-ups:

$$\{K_{x_0, r_k}\}, \quad K_{x_0} = \frac{K - x_0}{r_k}, \quad r_k \to 0.$$

By Lemma 1, $\Sigma(K)$ contains the half-line, that is,

$$\{\lambda x_0 : \lambda > 0\} \subset \Sigma(K)$$

We have:

$$\frac{\lambda x_0 - x_0}{r_k} \in \Sigma(K_{x_0, r_k}), \quad \forall \lambda > 0$$

$$\therefore \frac{(\lambda - 1)}{r_k} x_0 \in \Sigma(K_{x_0, r_k}) \quad \forall \lambda > 0.$$

Now, given any $\alpha > 0$, for each $k = 1, 2, 3, \ldots$, if $\lambda(k) = \alpha r_k + 1$, then $\frac{\lambda(k) - 1}{r_k} = \alpha$. Hence, for any $\lambda > 0$, $\lambda x_0 \in \Sigma(K_{x_0, r_k})$. Thus, from the Theorem in Page 36.2, Lecture 36, we obtain that:

$$\boxed{\{\lambda x_0 : \lambda > 0\} \subset \Sigma(\tilde{K})} \quad (4)$$

where K_2 is the limit of the blow-ups K_{x_0, r_k}; that is,

$$\boxed{K_{x_0, r_k} \xrightarrow{loc} \tilde{K}}, \quad \tilde{K} \text{ is a singular minimizing cone in } \mathbb{R}^n.$$

From (4):

$$\boxed{\mathcal{H}^1(\Sigma(\tilde{K})) = \infty} \quad (5)$$

WLOG $x_0 = e_n$. Now, for every $R>0$,

$$\int_{B_R \cap \partial^* K_{x_0,r_k}} |e_n \cdot \nu_{K_{x_0,r_k}}| \, d\mathcal{H}^{n-1} = \frac{1}{r_k^{n-1}} \int_{B(e_n, r_k R) \cap \partial^* K} |e_n \cdot \nu_K| \, d\mathcal{H}^{n-1}$$

$$\leq \frac{1}{r_k^{n-1}} \int_{B(e_n, r_k R)\cap \partial^* K} |y - e_n| \, d\mathcal{H}^{n-1} \; ;$$

since $\nu_K(y)\cdot y = 0$ $\Rightarrow$ $e_n \cdot \nu_K(y) = (e_n - y)\cdot \nu_K(y)$ for $\mathcal{H}^{n-1}$- a.e. $y \in \partial^* K$

$$\leq \frac{r_k R}{r_k^{n-1}} \int_{B(e_n, r_k R)\cap \partial^* K} d\mathcal{H}^{n-1} \; ;$$

because $y \in B(e_n, r_k R)$ $\Rightarrow$ $|y - e_n| \leq r_k R$

$$= \frac{r_k R}{r_k^{n-1}} P(K, B(e_n, r_k R))$$

$$\leq \frac{r_k R}{r_k^{n-1}} n\omega_n (r_k R)^{n-1} \; ;$$

Since for a perimeter minimizer E:

$$\omega_{n-1} \leq \frac{P(E; B(x,r))}{r^{n-1}} \leq n\omega_n$$

$\forall x \in \partial E \cap A$, $\forall\, 0 < r < \mathrm{dist}(x, \partial A)$.

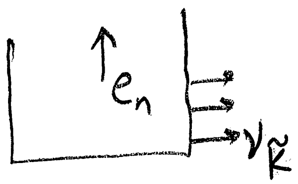
$$= C(n) R^n r_k$$

Since $\mu_{K_{x_0,r_k}} \xrightarrow{*} \mu_{\tilde{K}}$, by the Reshetnyak lower semicontinuity theorem we get:

$$\int_{B_R \cap \partial^* \tilde{K}} |e_n \cdot \nu_{\tilde{K}}| \, d\mathcal{H}^{n-1} \leq \liminf_{k\to\infty} \int_{B_R \cap \partial^* K_{x_0,r_k}} |e_n \cdot \nu_{K_{x_0,r_k}}| \, d\mathcal{H}^{n-1} = 0.$$

Hence:

$$\boxed{e_n \cdot \nu_{\tilde{K}} = 0 \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } y \in \partial^* \tilde{K}}$$



Hence, by Lemma 2, part (ii), $\tilde{K}$ is equivalent to a cylinder:

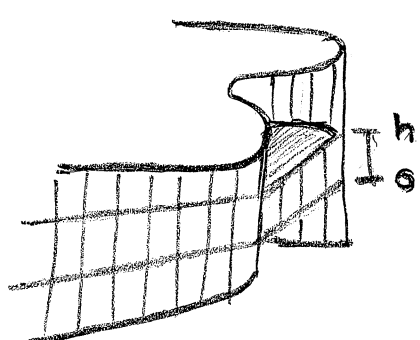
$$\boxed{F \times \mathbb{R}, \text{ where } F \text{ is a perimeter minimizer in } \mathbb{R}^{n-1}}$$

Indeed, F is minimal for otherwise, by contradiction $\exists H,\ \delta > 0$ s.t.

$$P(H) \le P(F) - \delta$$

Form a comparison set:

$$A = H \times [0,h] \cup (F \times [h, \infty)] \cup (F \times [-\infty, 0))$$



$$\Rightarrow \quad P(A) - P(\tilde{K}) \le 2C - \delta h < 0 \quad \text{if } h \gg 1. \ !$$

contra-diction

Hence F is minimal.

By Lemma 2, part (i):

$$\underset{\tilde{K}}{\underset{\parallel}{\partial^*(F \times \mathbb{R})}} = \partial^* F \times \mathbb{R}$$

$$\therefore \quad \Sigma(\tilde{K}) = \Sigma(F) \times \mathbb{R}$$

If $\Sigma(F) = \emptyset$ then $\Sigma(\tilde{K}) = \emptyset$, which is not possible due to (5).

Hence:

$$\mathcal{H}^0(\Sigma(F)) > 0$$

$\Rightarrow$ F is a singular minimizing cone in $\mathbb{R}^{n-1}$. ∎

A Bernstein-type theorem

<u>Theorem</u>: If E is a perimeter minimizer in $\mathbb{R}^n$, where $2 \le n \le 7$, then E is a half-space.

<u>Proof</u>: We fix $x \in \partial E$

Let $\{r_k\}$, $r_k \to 0^+$, $\{R_k\}$, $R_k \to \infty$.

Up to extracting subsequences, $\exists F_0, F_\infty$ s.t:

$$E_{x,r_k} \xrightarrow{loc} F_0, \quad E_{x,R_k} \xrightarrow{loc} F_\infty \quad \text{as } k \to \infty,$$

and F_0, F_∞ are perimeter minimizers in $\mathbb{R}^n$.

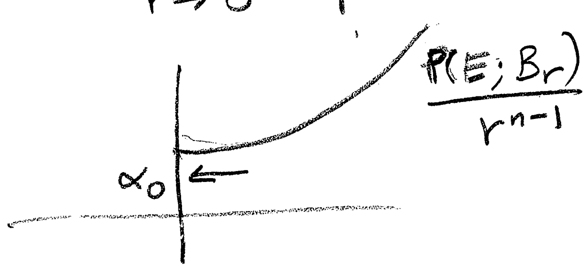
For a.e. $s>0$ such that $\mathcal{H}^{n-1}(\partial^* F_0 \cap \partial B_s) = 0$ we have

$$P(F_0; B_s) = \lim_{k\to\infty} P(E_{x,r_k}; B_s)$$

$$= \lim_{k\to\infty} \frac{1}{r_k^{n-1}} P(E; B_{sr_k})$$

$$= s^{n-1} \lim_{k\to\infty} \frac{P(E; B_{sr_k})}{(sr_k)^{n-1}}$$

$$= s^{n-1}\alpha_0 \; ; \quad \text{where } \alpha_0 = \lim_{r\to 0} \frac{P(E; B_r)}{r^{n-1}} \ge \omega_{n-1}$$



$$\therefore \frac{P(F_0; B_s)}{s^{n-1}} = \alpha_0 \quad \text{a.e. } s>0$$

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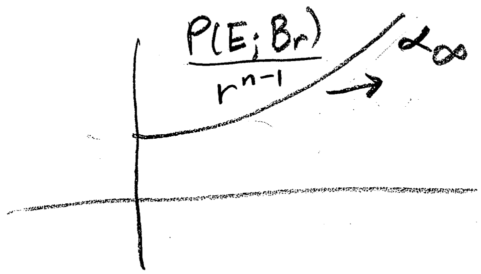
Similarly:

$$P(F_\infty; B_s) = \lim_{k\to\infty} P(E_{x,R_k}; B_s)$$

$$= \lim_{k\to\infty} \frac{1}{R_k^{n-1}} P(E; B_{sR_k})$$

$$= s^{n-1} \lim_{k\to\infty} \frac{P(E; sR_k)}{(sR_k)^{n-1}}$$

$$= s^{n-1}\alpha_\infty \; ; \quad \text{where } \alpha_\infty = \lim_{r\to\infty} \frac{P(E; B_r)}{r^{n-1}}$$



Hence we have:

$$\boxed{\frac{P(F_0; B_s)}{s^{n-1}} = \alpha_0 \quad \text{and} \quad \frac{P(F_\infty; B_s)}{s^{n-1}} = \alpha_\infty \, , \quad \text{a.e. } s>0.}$$

By the monotonicity formula:

$$\frac{d}{ds} \frac{P(F_0; B_s)}{s^{n-1}} = \frac{d}{ds}\left(\int_{B_s \cap \partial^* F_0} \frac{|\nu_{F_0}(y)\cdot y|^2}{|y|^{n+1}} \, d\mathcal{H}^{n-1}(y)\right) \quad \text{for a.e. } s$$

$\Rightarrow$ for a.e. $s_1 < s_2$:

$$\int_{\partial^* F_0 \cap (B_{s_2}\setminus B_{s_1})} \frac{|\nu_{F_0}(y)\cdot y|^2}{|y|^{n+1}} = \int_{s_1}^{s_2} \frac{d}{ds}\int_{\partial^* F_0 \cap B_s} \frac{|\nu_{F_0}(y)\cdot y|^2}{|y|^{n+1}} \, ds = \int_{s_1}^{s_2} \frac{d}{ds} \frac{P(F_0; B_s)}{s^{n-1}}$$

$$= \frac{P(F_0; B_{s_2})}{s_2^{n-1}} - \frac{P(F_0; B_{s_1})}{s_1^{n-1}} = 0$$

$$\Rightarrow \int_{\partial^* F_0 \cap (B_{s_2} \setminus B_{s_1})} \frac{|\nu_{F_0}(y) \cdot y|^2}{|y|^{n+1}} = 0 \quad \text{a.e. } s_1 < s_2$$

$$\Rightarrow \quad \nu_{F_0}(y) \cdot y = 0 \quad \mathcal{H}^{n-1}\text{-a.e. } y \in \partial^* F_0 \setminus \{0\}$$

$$\Rightarrow \quad \boxed{F_0 \text{ is a cone}}$$

Exactly in the same way we prove:

$$\boxed{F_\infty \text{ is a cone}}$$

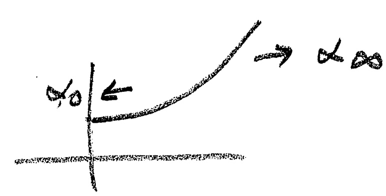
Since $2 \le n \le 7$ and F_0, F_∞ are cones, perimeter minimizers in $\mathbb{R}^n$, the by our main theorem on Regularity of singularities, part (i), we obtain:

$$\boxed{F_0,\ F_\infty \text{ are half-spaces}}$$

(Otherwise F_0, F_∞ would be perimeter minimizers in $\mathbb{R}^n$ with singularities, which is not possible if $2 \le n \le 7$).

Going back to E, by the monotonicity of density ratios:

$$\alpha_0 \le \frac{P(E; B_r)}{r^{n-1}} \le \alpha_\infty$$



But F_0, F_∞ are half-spaces $\Rightarrow \alpha_0 = \alpha_\infty = \omega_{n-1}$

Hence

$$\frac{P(E; B_r)}{r^{n-1}} = \omega_{n-1}, \quad \forall r > 0,$$

and again by the monotonicity formula:

$$\boxed{E \text{ is a cone}}$$

But since $2 \le n \le 7$, our main theorem on Analysis of singularities yields:

$$\boxed{E \text{ is a half-space}},$$

which is the desired result. ∎