Lecture 4

Recall :

Lusin's theorem : μ Borel measure on $\mathbb{R}^n$, $u : \mathbb{R}^n \to \mathbb{R}$ Borel function, $E \subset \mathbb{R}^n$ Borel, $\mu(E) < \infty$, then $\forall \varepsilon > 0$, $\exists K \subset E$ such that:

u is continuous on K and $\mu(E \setminus K) < \varepsilon$.

Using the method in the proof of Lusin's thm we get:

Theorem : If μ is a Radon measure on $\mathbb{R}^n$ and $p \in [1, \infty)$, then for every $u \in L^p(\mathbb{R}^n, \mu)$ there exits a sequence $\{u_k\} \subset C_c(\mathbb{R}^n)$ such that.

$$\lim_{k \to \infty} \int_{\mathbb{R}^n} |u - u_k|^p \, d\mu = 0$$

Riesz's theorem and vector-valued Radon measures.

Consider $C_c(\mathbb{R}^n)$, the space of continuous functions with compact support in $\mathbb{R}^n$, equiped with the following topology:

" $\varphi_i \to \varphi$ in $C_c(\mathbb{R}^n)$ if $\varphi_i \to \varphi$ uniformly on $\mathbb{R}^n$ and, for a compact set $K \subset \mathbb{R}^n$:

$$\operatorname{spt} \varphi \cup \left[\bigcup_{i=1}^{\infty} \operatorname{spt} \varphi_i \right] \subset K$$

Consider a linear functional $L: C_c(\mathbb{R}^n; \mathbb{R}^m) \to \mathbb{R}$. Then

<u>Lemma</u>: L is continuous with respect to convergence in $C_c(\mathbb{R}^n; \mathbb{R}^m)$ if and only if it is bounded, in the sense that, for every compact set $K \subset \mathbb{R}^n$,

$$\boxed{\sup\{\langle L, \varphi\rangle : \varphi \in C_c(\mathbb{R}^n; \mathbb{R}^m),\ \operatorname{spt}\varphi \subset K,\ |\varphi| \le 1\} < \infty \quad (*)}$$

<u>Ex</u>: Let μ be a Radon measure on $\mathbb{R}^n$, let $L: C_c(\mathbb{R}^n) \to \mathbb{R}$:

$$\langle L, \varphi\rangle = \int_{\mathbb{R}^n} \varphi \, d\mu, \quad \varphi \in C_c(\mathbb{R}^n)$$

It is easy to see that L is continuous on $C_c(\mathbb{R}^n)$, since $\varphi_i \to \varphi$ in $C_c(\mathbb{R}^n)$ implies:

$$\begin{aligned}
|\langle L, \varphi_i\rangle - \langle L, \varphi\rangle| &= \left| \int_{\mathbb{R}^n} \varphi_i \, d\mu - \int_{\mathbb{R}^n} \varphi \, d\mu \right| \\
&= \left| \int_{\mathbb{R}^n} (\varphi_i - \varphi) \, d\mu \right| \\
&\le \int_{\mathbb{R}^n} |\varphi_i - \varphi| \, d\mu \\
&\le \int_K \varepsilon \, d\mu, \quad \text{for } i \text{ large, since } \varphi_i \to \varphi \text{ in } C_c(\mathbb{R}^n). \\
&= \varepsilon \, \mu(K)
\end{aligned}$$

Clearly, $(*)$ is true too.

Ex : μ Radon measure on $\mathbb{R}^n$
$f \in L^1_{loc}(\mathbb{R}^n, \mu; \mathbb{R}^m)$.
Define $f\mu : C_c(\mathbb{R}^n; \mathbb{R}^m) \to \mathbb{R}$ as:

$$\langle f\mu, \varphi \rangle = \int_{\mathbb{R}^n} (\varphi \cdot f)\, d\mu, \quad \varphi \in C_c(\mathbb{R}^n; \mathbb{R}^m)$$

Riesz's theorem ensures that every bounded linear functional on $C_c(\mathbb{R}^n; \mathbb{R}^m)$ can be represented as a product $f\mu$.

<u>Def</u>: Let $L : C_c(\mathbb{R}^n; \mathbb{R}^n)$ bounded linear functional, define the <u>total variation</u> of L as a set function:

$$|L| : \mathcal{P}(\mathbb{R}^n) \to [0, \infty]$$

$$|L|(A) = \sup \{ \langle L, \varphi \rangle : \varphi \in C_c(A; \mathbb{R}^m),\ |\varphi| \le 1 \}, \quad \underset{\text{open}}{A \subset \mathbb{R}^n}$$

$$|L|(E) = \inf \{ |L|(A) : E \subset A,\ A \text{ open} \}, \quad E \subset \mathbb{R}^n \text{ arbitrary.}$$

<u>Riesz's theorem</u>: $L : C_c(\mathbb{R}^n; \mathbb{R}^m) \to \mathbb{R}$ bounded linear functional, then:

- $|L|$ is a Radon measure on $\mathbb{R}^n$.
- $\exists\, g : \mathbb{R}^n \to \mathbb{R}^m$, $|g| = 1$ $|L|$-a.e. on $\mathbb{R}^n$, $|L|$-measurable

and
$$\langle L, \varphi \rangle = \int_{\mathbb{R}^n} (\varphi \cdot g)\, d|L|, \quad \forall \varphi \in C_c(\mathbb{R}^n; \mathbb{R}^m)$$

4.4

Ex: If $L = f\mu$, μ Radon measure $\Rightarrow$
$|L| = |f|\mu$, $g = \frac{f}{|f|}$, $|f|$-μ-a.e. on $\mathbb{R}^n$

Remark: Note that the continuous linear functional $L: C_c(\mathbb{R}^n) \to \mathbb{R}$

$$\langle L, \varphi \rangle = \int_{\mathbb{R}^n} \varphi \, d\mu, \quad \varphi \in C_c(\mathbb{R}^n) \quad ; \ \mu \text{ Radon measure}$$

is positive, that is;

$$\langle L, \varphi \rangle \geq 0 \quad \text{if } \varphi \geq 0$$

or, equivalently, L is monotone, that is,

$$\langle L, \varphi_1 \rangle \leq \langle L, \varphi_2 \rangle, \quad \text{if } \varphi_1 \leq \varphi_2$$

Radon measures can be unambiguosly identified with monotone linear functionals on $C_c(\mathbb{R}^n)$,

Indeed:

L monotone $\Rightarrow$ L bounded (Exercise 4.16)

$$\Rightarrow \langle L, \varphi \rangle = \int_{\mathbb{R}^n} \varphi \, g \, d|L|, \quad |g| = 1, \ |L| \text{ a.e.}$$

$$\Rightarrow g \geq 0, \ |L|\text{-a.e, by Exercise 4.5}$$

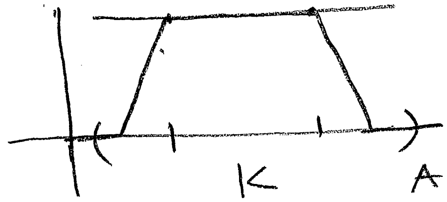
$$\Rightarrow g = 1, \ |L|\text{-a.e.}$$

$$\Rightarrow \langle L, \varphi \rangle = \int_{\mathbb{R}^n} \varphi \, d|L|, \quad \forall \varphi \in C_c(\mathbb{R}^n).$$

On the other hand:

$\int_{\mathbb{R}^n} \varphi \, d\mu_1 = \int_{\mathbb{R}^n} \varphi \, d\mu_2 \ \forall \varphi \in C_c(\mathbb{R}^n) \Rightarrow \mu_1 = \mu_2$, because:

if K is compact, A open $K \subset A$,
$\exists\, \varphi \in C_c(\mathbb{R}^n)$, $\chi_K \leq \varphi \leq \chi_A$.



$$\mu_1(K) = \int_{\mathbb{R}^n} \chi_K \, d\mu_1 \leq \int_{\mathbb{R}^n} \varphi \, d\mu_1 = \int_{\mathbb{R}^n} \varphi \, d\mu_2 \leq \mu_2(A)$$

$\Rightarrow \mu_1(E) \leq \mu_2(E)$, E Borel (by regularity of Radon measures)

In the same way:

$$\mu_2(E) \leq \mu_1(E) \Longrightarrow \mu_1 = \mu_2 \text{ on } \mathcal{B}(\mathbb{R}^n)$$

Now, μ_1 and μ_2 are Borel regular measures, and from Exercise 2.6 (μ, ν Borel regular measures on $\mathbb{R}^n$, $\mu = \nu$ on $\mathcal{B}(\mathbb{R}^n) \Rightarrow \mu = \nu$ on $\mathcal{P}(\mathbb{R}^n)$) we conclude $\mu_1 = \mu_2$ on $\mathcal{P}(\mathbb{R}^n)$.

<u>**Remark**</u>: $L : C_c(\mathbb{R}^n, \mathbb{R}^m) \to \mathbb{R}$ bounded linear functional allows to define a <u>vector-valued Radon measure</u> as follows

$$\nu(E) = \int_E g \, d|L| \; ; \quad \text{that is, } \nu = g|L|$$

$$\nu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \nu(E_i), \quad E_i \text{ disjoint}$$

$$\langle \nu, \varphi \rangle = \int_E (\varphi \cdot g) \, d|L|$$

Conversely, if ν is a vector-valued Radon measure the $\nu = g|\nu|$ (Polar decomposition) and $|g|=1$, $|\nu|$-a.e., and ν induces the linear functional:

$$\langle L, \varphi \rangle = \int_{\mathbb{R}^n} (\varphi \cdot g)\, |\nu| \,, \quad \varphi \in C_c(\mathbb{R}^n; \mathbb{R}^m).$$

Thus, we can write:

$$\boxed{C_c(\mathbb{R}^n; \mathbb{R}^m)^* = \{\mathbb{R}^m\text{-valued Radon measures on } \mathbb{R}^n\},}$$

When $m=1$, then ν is called a <u>signed Radon measure on $\mathbb{R}^n$</u>. In this case $\nu = g|\nu|$, with $|g|=1$, $|\nu|$-a.e, and

$$\left.\begin{aligned} \nu^+ &= \chi_{\{g=1\}}|\nu| = \frac{|\nu|+\nu}{2} \\ \nu^- &= \chi_{\{g=-1\}}|\nu| = \frac{|\nu|-\nu}{2} \end{aligned}\right\} \begin{aligned} \nu &= \nu^+ - \nu^- \\ |\nu| &= \nu^+ + \nu^- \end{aligned}$$

Jordan Decomp. of measures

Recall again; for $m=1$:

$$\boxed{\begin{gathered} C_c(\mathbb{R}^n)^* = \{\text{signed Radon measures on } \mathbb{R}^n\} \\ \text{and} \\ \left\{\begin{matrix}\text{Monotone linear} \\ \text{functionals on } C_c(\mathbb{R}^n)\end{matrix}\right\} = \left\{\begin{matrix}\text{Radon measures} \\ \text{on } \mathbb{R}^n\end{matrix}\right\} \end{gathered}}$$

Weak-star convergence

Definition: $\{\mu_i\}_{i=1}^{\infty}, \mu$ $\mathbb{R}^m$-valued Radon measures on $\mathbb{R}^n$

μ_i weak-star converges to μ, $\mu_i \xrightarrow{*} \mu$, if

$$\int_{\mathbb{R}^n} \varphi \cdot d\mu = \lim_{i\to\infty} \int_{\mathbb{R}^n} \varphi \cdot d\mu_i, \quad \varphi \in C_c(\mathbb{R}^n; \mathbb{R}^m)$$

Ex: $\{x_i\} \subset \mathbb{R}^n$, $\mu_i = \delta_{x_i}$, $x_i \to x_0$ then

$$\int_{\mathbb{R}^n} \varphi \, d\mu_i = \varphi(x_i) \to \varphi(x_0) = \int_{\mathbb{R}^n} \varphi \, d\mu, \quad \varphi \in C_c(\mathbb{R}^n)$$

with $\mu = \delta_{x_0} \Rightarrow \boxed{\delta_{x_i} \xrightarrow{*} \delta_{x_0}}$.

If $|x_i| \to \infty \Rightarrow \boxed{\delta_{x_i} \xrightarrow{*} 0}$.

Note: For every $\{x_i\} \subset \mathbb{R}^n$ $\exists \{x_{i_k}\}$ subsequence such that $\delta_{x_{i_k}} \xrightarrow{*} \delta_{x_0}$ $(x_0 \in \mathbb{R}^n)$ or $\delta_{x_{i_k}} \xrightarrow{*} 0$.

Ex (Spreading of mass): $\mu_k = \sum_{i=1}^{k} \frac{1}{k} \delta_{i/k} \xrightarrow{*} \mathcal{L}^1[0,1]$

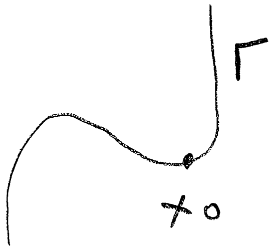
$$\int_{\mathbb{R}^n} \varphi \, d\mu_k = \sum_{i=1}^{k} \frac{1}{k} \varphi(i/k) \to \int_{(0,1)} \varphi \, dx$$

Ex (Concentration of mass): $\mu_k = \frac{1}{|B_k|} \chi_{B_k(0)} \xrightarrow{*} \delta_0$

Ex (Averaging effects): $\mu_k = (1 + \sin(kx)) \mathcal{L}^1 \xrightarrow{*} \mathcal{L}^1$

A fundamental idea in Geometric Measure Theory (Blow-ups):

"Existence of tangent spaces in terms of weak-star convergence of Radon measures",



Γ smooth curve in $\mathbb{R}^n$

$\Gamma = \gamma((a,b))$, $\gamma : (a,b) \to \mathbb{R}^n$ injective

$\gamma(t_0) = x_0$

Tangent space to Γ at x_0 is line $\Pi = \{s\gamma'(t_0) : s \in \mathbb{R}\}$

- Consider Γ as the Radon measure $\mu = \mathcal{H}^1 \llcorner \Gamma$
- Define the blow-ups $\mu_{x_0,r}$ of μ at x_0 as:

$$\mu_{x_0,r} = \frac{1}{r}(\Phi_{x_0,r})_{\#}(\mathcal{H}^1 \llcorner \Gamma) = \mathcal{H}^1 \llcorner \left(\frac{\Gamma - x_0}{r}\right)$$

$$\Phi_{x_0,r}(y) = \frac{y - x_0}{r}, \quad y \in \mathbb{R}^n$$

$$\Phi_{x_0,r} : B_r(x) \to B_1(0) \quad \text{(blow-up)}$$

<u>Claim</u>: $\mu_{x_0,r} \xrightarrow{*} \mathcal{H}^1 \llcorner \Pi$ as $r \to 0^+$.

Indeed, if $\varphi \in C_c(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \varphi \, d\mu_{x_0,r} = \frac{1}{r}\int_{\mathbb{R}^n} \varphi \, d(\Phi_{x_0,r})_{\#}$$

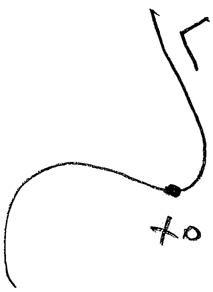
$$= \frac{1}{r}\int_{\mathbb{R}^n} \varphi \circ \Phi_{x_0,r} \, d\mu$$

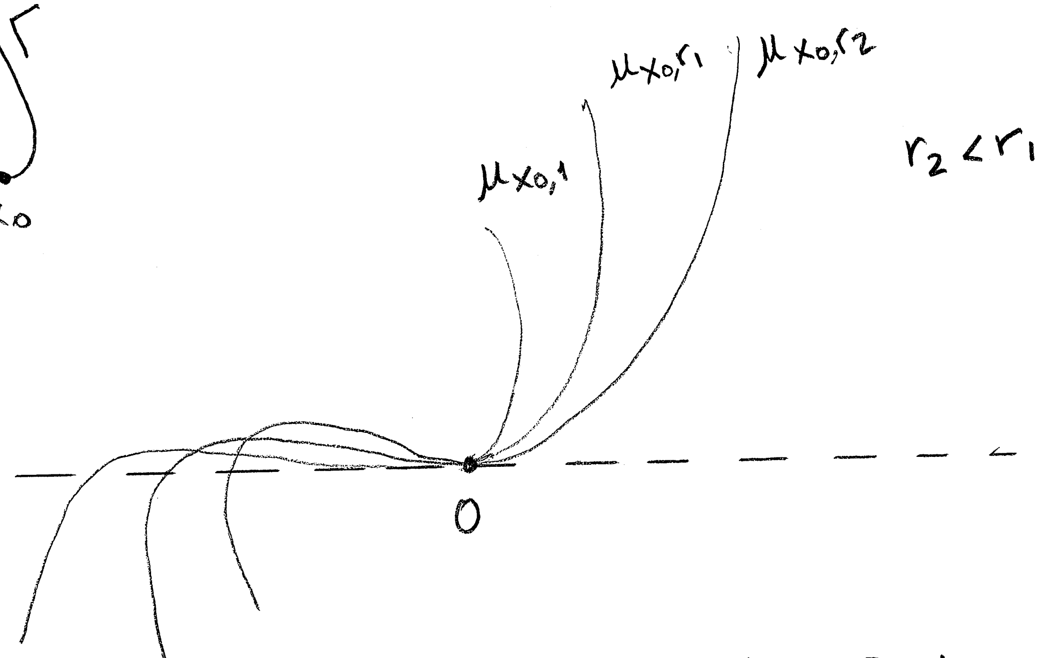
$$= \frac{1}{r}\int_\Gamma \varphi\left(\frac{y-x_0}{r}\right) d\mathcal{H}^1(y)$$

$$= \frac{1}{r}\int_a^b \varphi\left(\frac{\gamma(t)-\gamma(t_0)}{r}\right)|\gamma'(t)|\,dt$$

$$\underset{t=t_0+rs}{=} \int_{-\left(\frac{t_0-a}{r}\right)}^{\frac{b-t_0}{r}} \varphi\left(\frac{\gamma(t_0+rs)-\gamma(t_0)}{r}\right)|\gamma'(t_0+rs)|\,ds$$

$$\longrightarrow \int_{\mathbb{R}} \varphi(s\gamma'(t_0))\,|\gamma'(t_0)|\,ds = \int_\Pi \varphi\,d\mathcal{H}^1, \quad \text{as } r\to 0^+$$





The blow-ups at x_0 of the Radon measure $\mathcal{H}^1 \llcorner \Gamma$ weak-star converge to $\mathcal{H}^1 \llcorner \Pi$.

Prop: $\{\mu_i\}$, μ Radon measures on $\mathbb{R}^n$ (recall non-negatives). The following are equivalent:

(i) $\mu_i \xrightarrow{*} \mu$

(ii) $\forall K$ compact, $\limsup_{i\to\infty} \mu_i(K) \le \mu(K)$

$\forall A$ open, $\liminf_{i\to\infty} \mu_i(A) \ge \mu(A)$

(iii) $\forall E$ Borel, $\mu(\partial E) = 0 \Rightarrow \mu_i(E) \to \mu(E)$

Prop: $\{\mu_i\}$ Vector-valued Radon measures. Then

(i) $\mu_i \xrightarrow{*} \mu \Rightarrow \forall A$ open $|\mu|(A) \le \liminf_{i\to\infty} |\mu_i(A)|$

Recall from Riesz's Theorem:

$|\mu|(A) = \sup\left\{ \int_{\mathbb{R}^n} \varphi \cdot d\mu : \varphi \in C_c(A;\mathbb{R}^m),\ |\varphi| \le 1 \right\}$

(ii) If $\mu_i \xrightarrow{*} \mu$ and $|\mu_i| \xrightarrow{*} \nu \Rightarrow |\mu|(E) \le \nu(E)$, $\forall E$ Borel

E Borel $\nu(\partial E) = 0 \Rightarrow \mu_i(E) \to \mu(E)$

Weak-star compactness criteria

Compactness for Radon measures: $\{\mu_i\}$ Radon measures on $\mathbb{R}^n$ such that $\forall K \subset \mathbb{R}^n$ compact:

$$\sup_i \mu_i(K) < \infty$$

$\Rightarrow \exists \mu$ Radon on $\mathbb{R}^n$ and $\{i_k\}_{k=1}^\infty$ s.t. $\mu_{i_k} \xrightarrow{*} \mu$

Corollary: $\{\mu_i\}$ are $\mathbb{R}^m$-valued Radon measures on $\mathbb{R}^n$,

$\sup_i |\mu_i|(K) < \infty$, $\forall K \subset \mathbb{R}^n$ compact

$\Rightarrow \exists \mu$ $\mathbb{R}^m$-valued and $\{i_k\}_{k=1}^\infty$ s.t. $\mu_{i_k} \xrightarrow{*} \mu$ as $k \to \infty$.

Ex: $\mu_k = \delta_{1/k} - \delta_{-\frac{1}{k}} \overset{*}{\rightharpoonup} 0$ and

$|\mu_k| = \delta_{1/k} + \delta_{-1/k} \overset{*}{\rightharpoonup} 2\delta_0$, thus it is not necessarily true that $\mu_k^+ \overset{*}{\rightharpoonup} \mu^+$ or $\mu_k^- \overset{*}{\rightharpoonup} \mu^-$.

Regularization of Radon measures

We can take convolutions of Radon measures: Let μ be an $\mathbb{R}^m$-valued Radon measure on $\mathbb{R}^n$, define

$$(\mu * \rho_\varepsilon)(x) = \int_{\mathbb{R}^n} \rho_\varepsilon(x-y)\, d\mu(y), \quad x \in \mathbb{R}^n$$

Then:

$$\mu_\varepsilon := \mu * \rho_\varepsilon \in C^\infty(\mathbb{R}^n, \mathbb{R}^m)$$

μ_ε is an $\mathbb{R}^m$-valued Radon measure on $\mathbb{R}^n$ given by:

$$\langle \mu_\varepsilon, \varphi \rangle = \int_{\mathbb{R}^n} \varphi(x) \cdot (\mu * \rho_\varepsilon)(x)\, dx, \quad \varphi \in C_c(\mathbb{R}^n; \mathbb{R}^m)$$

$$\mu_\varepsilon(E) = \int_E (\mu * \rho_\varepsilon)(x)\, dx$$

This is a useful theorem:

Theorem: $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu$, $|\mu_\varepsilon| \overset{*}{\rightharpoonup} |\mu|$