Lection 5

Besicovitch's covering theorem.

<u>Theorem</u>: $n \geq 1$. Then $\exists c(n)$ such that:
If $\mathcal{F}$ is a family of closed non-degenerate balls on $\mathbb{R}^n$, and either the set $\mathcal{C}$ of the centers of the balls in $\mathcal{F}$ is bounded or

$$\sup\{\operatorname{diam}(\bar{B}) : \bar{B} \in \mathcal{F}\} < \infty,$$

then $\exists \mathcal{F}_1, \ldots, \mathcal{F}_{c(n)}$ (possibly empty) subfamilies of $\mathcal{F}$ such that:
(i) Each $\mathcal{F}_i$ is disjoint and at most countable
(ii) $\mathcal{C} \subset \bigcup_{i=1}^{c(n)} \bigcup_{\bar{B} \in \mathcal{F}_i} \bar{B}$

<u>Note</u>: For every $x \in \mathbb{R}^n$, the number of balls $\bar{B}$ in the collection $\bigcup_{i=1}^{c(n)} \bigcup_{\bar{B} \in \mathcal{F}_i} \bar{B}$ such that $x \in \bar{B}$ is less or equal than $c(n)$.

<u>Idea of Proof</u>:
If $\bar{B}_1$ is a ball of "max diameter", get rid of all balls whose centers are in $\bar{B}_1$; that is: $\exists \bar{B}_1 \in \mathcal{F}$ with:

$$\operatorname{diam}(\bar{B}_1) \geq \frac{2}{3} \sup\{\operatorname{diam}(\bar{B}) : \bar{B} \in \mathcal{F}\}$$

Let $\overline{B}_2$ be any ball from $\mathcal{F}$ whose center does not lie in $\overline{B}_1$ and such that:

$$\operatorname{diam} \overline{B}_2 \geq \frac{2}{3} \sup \{\operatorname{diam}(\overline{B}) : \overline{B} \in \mathcal{F}, \text{ the center of } \overline{B} \text{ is not in } \overline{B}_1\}$$

Let $\overline{B}_3$ be any ball from $\mathcal{F}$ whose center does not lie in $\overline{B}_1 \cup \overline{B}_2$ and such that:

$$\operatorname{diam} \overline{B}_3 \geq \frac{2}{3} \sup \{\operatorname{diam}(\overline{B}) : \overline{B} \in \mathcal{F}, \text{ and the center of } \overline{B} \text{ is not in } \overline{B}_1 \cup \overline{B}_2\}.$$

We continue like this. If this procedure stops after K steps, then we set $M = K$; otherwise, we set $M = \infty$. Let

$$G = \bigcup_{K=1}^{\infty} \overline{B}_K \,; \quad \text{where } \overline{B}_K := \overline{B}_K(x_K, r_K)$$

By construction, we have that:

$$|x_K - x_i| > r_i \,; \quad r_K \leq \frac{3}{2} r_i,$$

whenever $1 \leq i < K < M$ (which means that the center x_K is not in any of the balls $\overline{B}_i$).

It is proven in <u>Lemma 5.4</u> that:

$$\# \{K : 1 < K < N, \ \overline{B}_K \cap \overline{B}_N \neq \emptyset\} \leq \alpha(n) \quad \forall N < M$$

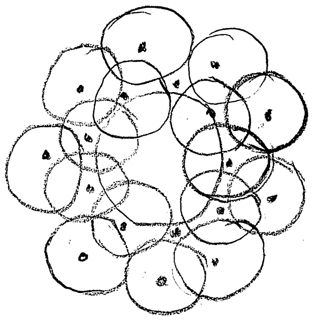

Figure 1.

Then:

(a) $\mathcal{C}$ is covered by $\mathcal{G}$

(b) $\mathcal{G}$ can be divided in to $c(n) := \alpha(n)+1$ subfamilies $\mathcal{F}_i$, where each $\mathcal{F}_i$ is disjoint

The proof of Lemma 5.4 is based on a geometric property of $\mathbb{R}^n$ and arragements of balls with particular radius that don't contain the centers of the other balls (Figure 1).

Remark: Let

$$\mathcal{F} = \left\{ B_k = B_{k+\frac{1}{k}}(ke_1) \right\}_{k=1}^{\infty}$$

The previous theorem is false for this $\mathcal{F}$ (since $\sup(\operatorname{diam} B_k) = \infty$). Indeed, in order to cover the centers $\mathcal{C} = \{ke_1\}_{k=1}^{\infty}$ we need infinitely many balls, but the origin belongs to all of them.

Corollary (Vitali's property): If μ is a Radon measure on $\mathbb{R}^n$, $\mathcal{F}$ is a family of closed non-degenerate balls whose set of centers $\mathcal{C}$ is bounded and μ-measurable, and, for every $x \in \mathcal{C}$:

$$\inf \{ \operatorname{diam}(\bar{B}) : \bar{B} \in \mathcal{F}, \ \bar{B} \text{ has center in } x \} = 0$$

then $\exists \{\bar{B}_i\}$ countable disjoint subfamily such that:

$$\mu\left(\mathcal{C} \setminus \bigcup_{i=1}^{\infty} \bar{B}_i\right) = 0$$

Idea of Proof:

By Besicovitch's covering theorem $\exists\ \mathcal{F}_1, \dots, \mathcal{F}_{c(n)}$ such that:

$$\mu(\mathcal{C}) \leq \sum_{i=1}^{c(n)} \sum_{\bar{B} \in \mathcal{F}_i} \mu(\mathcal{C} \cap \bar{B})$$

$$\begin{array}{lll} i=1 & \bar{B}_1^1, & \bar{B}_1^2, \dots \\ i=2 & \bar{B}_2^1, & \bar{B}_2^2 \dots \\ \vdots & \vdots & \\ i=c(n) & \bar{B}_{c(n)}^1, & \bar{B}_{c(n)}^2 \dots \end{array}$$

$\Rightarrow \exists\ i \in \{1, \dots, c(n)\}$ that maximizes $\sum_{\bar{B} \in \mathcal{F}_i} \mu(\mathcal{C} \cap \bar{B})$

Let $G = \mathcal{F}_i$, and thus.

$$\mu(\mathcal{C}) \leq c(n) \sum_{i=1}^{\infty} \mu(\mathcal{C} \cap \bar{B}_i), \quad \bar{B}_i \in G$$

$$\Rightarrow \mu(\mathcal{C}) \leq c(n)\, \mu\left(\mathcal{C} \cap \left(\bigcup\{\bar{B} : \bar{B} \in G\}\right)\right);$$

since the balls in the collection G are disjoint.

Now, $\exists N_1$ such that (since $\mu(\mathcal{C}) < \infty$):

$$\mu\left(\mathcal{C} \cap \bigcup_{i=1}^{N_1} \bar{B}_i\right) \geq \frac{\mu(\mathcal{C})}{2c(n)}$$

$$\Rightarrow \mu\left(\mathcal{C} \setminus \bigcup_{i=1}^{N_1} \bar{B}_i\right) \leq \theta\, \mu(\mathcal{C}) \qquad \theta = 1 - \frac{1}{2c(n)}$$

Then <u>Iterate</u> to obtain a sequence $\{N_k\}_{k=1}^{\infty}$ and a sequence $\mathcal{C}_k \subset \mathcal{C}$ and a countable family of closed balls $\{\bar{B}_j\}_{j=1}^{\infty} \subset \mathcal{F}$ with

$$\mathcal{C}_k = \mathcal{C} \setminus \bigcup_{j=1}^{N_k} \bar{B}_j, \quad \mu(\mathcal{C}_k) \leq \theta^k \mu(\mathcal{C})$$

Since $\mu(\mathcal{C}) < \infty \Rightarrow \mu\left(\mathcal{C} \setminus \bigcup_{j=1}^{\infty} \bar{B}_j\right) = 0.$ ∎

Lebesgue–Besicovitch Differentiation theorem:

μ, ν Radon measures on $\mathbb{R}^n$.

upper density: $D^+_\mu \nu(x) = \limsup_{r \to 0} \frac{\nu(\bar{B}(x,r))}{\mu(\bar{B}(x,r))}$, $\forall x \in \mathrm{Supp}(\mu)$

lower density: $D^-_\mu \nu(x) = \liminf_{r \to 0} \frac{\nu(\bar{B}(x,r))}{\mu(\bar{B}(x,r))}$

If $D^+_\mu \nu(x) = D^-_\mu \nu(x) \Rightarrow D_\mu \nu(x) := D^+_\mu \nu(x)$

$D_\mu \nu(x)$ is the density at x of ν w.r.t. μ

<u>Remark</u>: Recall a previous remark on "Foliations of Borel sets". Thus, for every $x \in \mathbb{R}^n$ $\exists$ at most countable many values of $r > 0$ such that either $\mu(\partial B(x,r)) > 0$ or $\nu(\partial B(x,r)) > 0$. Thus, if $D_\mu \nu$ is defined at x, we have

$$D_\mu \nu(x) = \lim_{r \to 0^+} \frac{\nu(B(x,r))}{\mu(B(x,r))}$$

So in evaluating $D_\mu \nu$ we can use open or closed balls. In the next theorem, closed balls are used in $D_\mu \nu(x)$ because its proof will use Vitali's property; and in the proof of Vitali's property, we can not use open balls instead of closed balls (see book by Ambrosio–Nicola–Fusco, Example 2.20).

<u>Theorem</u>: μ, ν Radon measures on $\mathbb{R}^n$.

Then:

$D_\mu \nu$ exists (finite) μ-a.e.

$D_\mu \nu$ is a Borel function, $D_\mu \nu \in L^1_{loc}(\mu)$

Moreover, $\nu = (D_\mu \nu)\mu + \nu_\mu^s$, and $\nu_\mu^s \perp \mu$

[ν_μ^s is concentrated on $(\mathbb{R}^n \setminus \operatorname{supp}\mu) \cup \{x \in \operatorname{supp}\mu : D_\mu^+ \nu = \infty\}$]

<u>Idea of Proof</u>: Uses Besicovitch covering theorem. For example, let us prove that $D_\mu \nu(x)$ is finite for μ-a.e. $x \in \mathbb{R}^n$.

Claim: $t \in (0,\infty)$, E bounded Borel set in $\mathbb{R}^n$, then

$$E \subset \{D_\mu^- \nu \leq t\} \Rightarrow \nu(E) \leq t\mu(E) \quad (1)$$

$$E \subset \{D_\mu^+ \nu \geq t\} \Rightarrow \nu(E) \geq t\mu(E) \quad (2)$$

Fix $\varepsilon > 0$, $\exists A$, $E \subset A$, $\mu(A) \leq \mu(E) + \varepsilon$. For (1), if $E \subset \{D_\mu^- \nu \leq t\} \Rightarrow$

$$\mathcal{F} = \{\bar{B}(x,r) : x \in E,\ \bar{B}(x,r) \subset A,\ \nu(\bar{B}(x,r)) \leq (t+\varepsilon)\mu(\bar{B}(x,r))\}$$

$\mathcal{F}$ satisfies the assumptions of Vitali's property $\Rightarrow \exists \{\bar{B}(x_k, r_k)\}$ a countable disjoint subfamily such that:

$$\nu\left(E \setminus \bigcup_{k=1}^{\infty} \bar{B}(x_k, r_k)\right) = 0, \text{ and}$$

$$\nu(E) = \sum_{k=1}^{\infty} \nu(\bar{B}(x_k, r_k)) \leq (t+\varepsilon) \sum_{k=1}^{\infty} \mu(\bar{B}(x_k, r_k))$$

$$\leq (t+\varepsilon)\mu(A) \leq (t+\varepsilon)(\mu(E)+\varepsilon)$$

$\varepsilon \to 0 \Rightarrow \nu(E) \leq t\mu(E)$. Similar proof for (2).

Now,
Let $Z = \{D_\mu^+ \nu = \infty\}$, $Z_{p,q} = \{D_\mu^- \nu < q < p < D_\mu^+ \nu\}$, $p, q \in \mathbb{Q}$.

We need to show that $\mu(Z) = \mu(Z_{p,q}) = 0$

Indeed, $Z \subset \{D_\mu^+ \nu \geq t\}$ $\forall t > 0$, and thus,

$$\mu(Z \cap B_R) \geq \frac{1}{t}\nu(Z \cap B_R) \leq \frac{\nu(B_R)}{t}$$

$$t \mapsto \infty,\ R \to \infty,\ \Rightarrow \mu(Z) = 0$$

Similarly, $\mu(Z_{p,q}) = 0 \Rightarrow D_\mu \nu$ is finite μ-a.e.

Lebesgue points

(Lebesgue points theorem): μ Radon on $\mathbb{R}^n$, $p \in [1, \infty)$, $u \in L^p_{loc}(\mathbb{R}^n, \mu)$, then for μ-a.e. $x \in \mathbb{R}^n$,

$$\lim_{r \to 0^+} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |u(x) - u(y)|^p \, d\mu(y) = 0$$

In this case, we say that x is a Lebesgue point of u with respect to μ.

Definition: Let $E \subset \mathbb{R}^n$, $x \in \mathbb{R}^n$, define;

$$\theta_n(E)(x) = \lim_{r \to 0^+} \frac{|E \cap B(x,r)|}{\omega_n r^n},$$

if the limit exists, $\theta_n(E)(x)$ is "the n-dimensional density of E at x".

Remark: Let $E \subset \mathbb{R}^n$ Lebesgue measurable, let $\mu = \mathcal{L}^n$. Then, by Lebesgue theorem:

$$\lim_{r\to 0^+} \frac{\int_{B(x,r)} \chi_E \, d\mathcal{L}^n}{|B(x,r)|} = \chi_E(x), \text{ for } \mathcal{L}^n\text{-a.e. } x$$

$$(*) \Rightarrow \frac{|E \cap B(x,r)|}{|B(x,r)|} = \begin{cases} 1, & \text{for } \mathcal{L}^n\text{-a.e. } x \in E \\ 0, & \text{for } \mathcal{L}^n\text{-a.e. } x \in \mathbb{R}^n \setminus E \end{cases}$$

Def: Given $t \in [0,1]$, the set of points of density t of E is defined as:

$$E^{(t)} = \{x \in \mathbb{R}^n : \theta_n(E)(x) = t\}.$$

Thus we have, by $(*)$:

$$|E \,\Delta\, E^{(1)}| = 0, \quad |(\mathbb{R}^n \setminus E) \,\Delta\, E^{(0)}| = 0.$$

"Every Lebesgue measurable set is Lebesgue equivalent to the set of its points of density one".

Proof of Lebesgue points theorem:

Let $\nu = u\mu$ (i.e. $\nu(E) = \int_E u \, d\mu$). Thus, For μ-a.e. $x \in \mathbb{R}^n$,

$$D_\mu \nu(x) = \lim_{r \to 0^+} \frac{\nu(B(x,r))}{\mu(B(x,r))}$$

$$= \lim_{r \to 0^+} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u \, d\mu$$

exists.

For every Borel set $E \subset \mathbb{R}^n$, since $\nu = (D_\mu \nu)\mu + \underbrace{\nu^s}_{=0}$:

$$\nu(E) = \int_E D_\mu \nu \, d\mu$$

$$\| $$

$$\int_E u \, d\mu$$

$\therefore \quad u = D_\mu \nu \quad \mu$-a.e. on $\mathbb{R}^n$.

$$\Rightarrow \lim_{r \to 0^+} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u \, d\mu = u(x), \text{ for } \mu\text{-a.e. } x$$

From here, it is easy to conclude:

$$\lim_{r \to 0^+} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |u(x) - u| \, d\mu = 0.$$