Two further applications of differentiation of measures

Campanato's criterion

This a basic tool in the regularity theory for variational problems, as it characterizes Hölder continuity in terms of the uniform decay of certain integral averages

Theorem (Campanato's criterion): $n \geq 1$, $p \in [1,\infty)$, $\gamma \in (0,1]$. Then $\exists\, C(n,p,\gamma)$ such that:

If $u \in L^p(B)$

$$(u)_{x,r} = \frac{1}{|B \cap B(x,r)|} \int_{B \cap B(x,r)} u \, dx \,, \quad x \in B,\ r > 0,$$

B is an open ball

and there exists constants α such that the Uniform decay condition

$$\left(\frac{1}{r^n} \int_{B \cap B(x,r)} |u - (u)_{x,r}|^p \right)^{1/p} \leq \alpha r^\gamma, \quad \forall x \in B$$

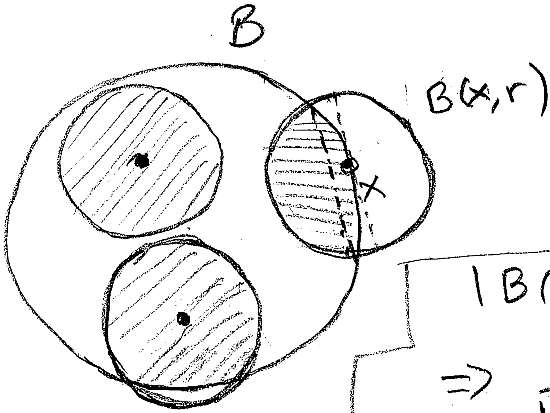
holds true, then there exists a function $\bar{u} : B \to \mathbb{R}$, $\bar{u} = u$ a.e. on B and

$$|\bar{u}(x) - \bar{u}(y)| \leq \tilde{\alpha} |x-y|^\gamma \quad \forall x, y \in B,$$

where $\tilde{\alpha} = C(n,p,\gamma)\alpha$.

Remark: Note that $\exists c(n) > 0$ such that: (6.2)

$$c(n) r^n \le |B \cap B(x,r)| \le \omega_n r^n$$



$$|B \cap B(x,r)| \ge c(n) r^n$$

$$\Rightarrow \frac{1}{|B \cap B(x,r)|} \le \frac{1}{c(n) r^n} \quad (*)$$

Also: $\frac{1}{\omega_n r^n} \le \frac{1}{|B \cap B(x,r)|}$

Then:

$$\lim_{r \to 0^+} \frac{1}{r^n} \int_{B \cap B(x,r)} |u - (u)_{x,r}|^p \, dy$$

$$\le \lim_{r \to 0^+} \frac{\omega_n}{|B(x,r)|} \int_{B \cap B(x,r)} |u - (u)_{x,r}|^p \, dy$$

$$= \lim_{r \to 0^+} \frac{\omega_n}{|B(x,r)|} \int_{B \cap B(x,r)} |u(y) - u(x) + u(x) - (u)_{x,r}|^p \, dy$$

$$\le \lim_{r \to 0^+} \left(\frac{2^{p-1} \omega_n}{|B(x,r)|} \int_{B \cap B(x,r)} |u(y) - u(x)|^p \, dy + \frac{2^{p-1} \omega_n}{|B(x,r)|} \int_{B \cap B(x,r)} |u(x) - (u)_{x,r}|^p \, dy \right)$$

(since $|a+b|^p \le 2^{p-1} (|a|^p + |b|^p)$).

$$= \lim_{r \to 0^+} \frac{2^{p-1} \omega_n}{|B(x,r)|} \int_{B(x,r)} |\chi_B u(y) - \chi_B u(x)|^p \, dy$$

$$+ \lim_{r \to 0^+} \frac{2^{p-1} \omega_n}{|B(x,r)|} |u(x) - (u)_{x,r}|^p \cdot |B \cap B(x,r)|$$

Hence:

$$\lim_{r\to 0^+} \frac{1}{r^n} \cdot \int_{B\cap B(x,r)} |u-(u)_{x,r}|^p dy$$

$$\leq 0 + \lim_{r\to 0^+} |u(x)-(u)_{x,r}|^p \quad ; \text{ for } \mathcal{L}^n\text{-a.e. } x \text{ that is a Lebesgue point for } \chi_B u, \text{ and } (*)$$

$$= \left| \lim_{r\to 0^+} \left(u(x) - (u)_{x,r}\right)\right|^p$$

$$= \left| \lim_{r\to 0^+} \left(u(x) - \frac{1}{|B\cap B(x,r)|}\int_{B\cap B(x,r)} u(y)\,dy\right)\right|^p = 0,$$

because for r small enough, $B\cap B(x,r) = B(x,r) \Rightarrow$

$$\lim_{r\to 0^+} \frac{1}{|B\cap B(x,r)|}\int_{B\cap B(x,r)} u(y)\,dy = \lim_{r\to 0^+} \frac{1}{|B(x,r)|}\int_{B(x,r)} u(y)\,dy \quad ; \text{ by } (*)$$

$$= u(x) \quad \text{for } \mathcal{L}^n\text{-a.e. } x\in B, \text{ Lebesgue point of } u$$

$(***)$

Therefore, we conclude:

$$\boxed{\lim_{r\to 0} \frac{1}{r^n}\int_{B\cap B(x,r)} |u-(u)_{x,r}|^p dy = 0 \quad \text{for } \mathcal{L}^n\text{-a.e. } x \text{ Lebesgue point of } \chi_B u \qquad (**)}$$

Proof of Campanato's Criterion:

Let

$$v_r(x) = (u)_{x,r}$$

For $r < R$ and $x \in B$ we have:

$$|v_r(x) - v_R(x)|^p = |v_r(x) - u(y) + u(y) - v_R(x)|^p \quad \forall y \in B$$

$$\leq 2^{p-1}\left(|v_r(x) - u(y)|^p + |u(y) - v_R(x)|^p\right)$$

Integrate both sides:

$$\int_{B\cap B(x,r)} |v_r(x) - v_R(x)|^p dy \leq 2^{p-1}\left(\int_{B\cap B(x,r)} |v_r(x) - u(y)|^p dy + \int_{B\cap B(x,r)} |u(y) - v_R(x)|^p dy\right)$$

$$\therefore \quad |v_r(x) - v_R(x)|^p |B\cap B(x,r)| \leq 2^{p-1}\left(\int_{B\cap B(x,r)} |v_r(x) - u(y)|^p dy + \int_{B\cap B(x,r)} |u(y) - v_R(x)|^p dy\right)$$

By (✱), $|B\cap B(x,r)| \geq c(n) r^n \Rightarrow$

$$c(n) r^n |v_r(x) - v_R(x)|^p \leq 2^{p-1}\left(\int_{B\cap B(x,r)} |v_r(x) - u(y)|^p dy + \int_{B\cap B(x,r)} |u(y) - v_R(x)|^p dy\right)$$

$$|v_r(x) - v_R(x)|^p \leq \frac{2^{p-1}}{c(n)}\left(\frac{1}{r^n}\int_{B\cap B(x,r)} |v_r(x) - u(y)|^p dy + \frac{1}{r^n}\int_{B\cap B(x,r)} |u(y) - v_R(x)|^p dy\right)$$

$\Rightarrow$

$$|v_r(x) - v_R(x)|^p \le \frac{2^{p-1}}{c(n)}\left(\frac{1}{r^n}\int_{B\cap B(x,r)} |v_r(x) - u(y)|^p dy + \frac{R^n}{r^n}\cdot\frac{1}{R^n}\int_{B\cap B(x,R)} |u(y) - v_R(x)|^p dy\right)$$

$$\le \frac{2^{p-1}}{c(n)}\left(\alpha^p r^{\gamma p} + \left(\frac{R}{r}\right)^n \alpha^p R^{\gamma p}\right),$$

$$\le \frac{2^{p-1}}{c(n)}\left[\left(\frac{R}{r}\right)^n \alpha^p R^{\gamma p} + \left(\frac{R}{r}\right)^n \alpha^p R^{\gamma p}\right]; \quad \text{since } \frac{R}{r} > 1$$

$$\le \frac{2^{p-1}}{c(n)}\cdot 2\cdot\left(\frac{R}{r}\right)^n \alpha^p R^{\gamma p};$$

$$= \frac{2^p}{c(n)}\left(\frac{R}{r}\right)^n \alpha^p R^{\gamma p};$$

$$\Rightarrow \boxed{|v_r(x) - v_R(x)| \le c(n,p)\left(\frac{R}{r}\right)^{n/p} \alpha R^{\gamma}}, \quad \forall r < R,\ x \in B$$

Set $r_0 = r$, $r_1 = \frac{r}{2}$, $r_2 = \frac{1}{2^2}r$, $r_3 = \frac{1}{2^3}r$, $\ldots$ $r_k = \frac{1}{2^k}r$, $\ldots$

$$\{r_k\}, \quad r_k \to 0, \quad r_{j+1} < r_j$$

Let $0 \le i < k$:

$$|v_{r_k}(x) - v_{r_i}(x)| \le \sum_{j=i}^{k-1} |v_{r_{j+1}} - v_{r_j}(x)| \le c(n,p)\alpha \sum_{j=i}^{k-1}\left(\frac{r_j}{r_{j+1}}\right)^{n/p}(r_j)^{\gamma}$$

$$= c(n,p)\alpha \sum_{j=i}^{k-1} 2^{n/p}\frac{r^{\gamma}}{2^{j\gamma}}$$

$\Rightarrow$ (6.6)

$$|v_{r_k}(x) - v_{r_i}(x)| = C(n,p)\,\alpha \sum_{j=i}^{k-1} 2^{n/p} \left(\frac{1}{2^\gamma}\right)^j r^\gamma$$

$$= C(n,p)\,\alpha \sum_{j=i}^{k-1} \left(\frac{1}{2^\gamma}\right)^j r^\gamma, \quad \forall x \in B \rightarrow (E)$$

$$\leq C(n,p,\gamma)\,\alpha\, r^\gamma.$$

$$\Rightarrow \boxed{|v_{r_k}(x) - v_{r_i}(x)| \leq C(n,p,\gamma)\,\alpha\, r^\gamma} \quad x \in B \quad \forall i,k \quad k > i \geq 0$$

With $i = 0$ (i.e $v_{r_i}(x) = v_r(x)$) we have:

$$|v_{r_k}(x) - v_r(x)| \leq C(n,p,\gamma)\,\alpha\, r^\gamma$$

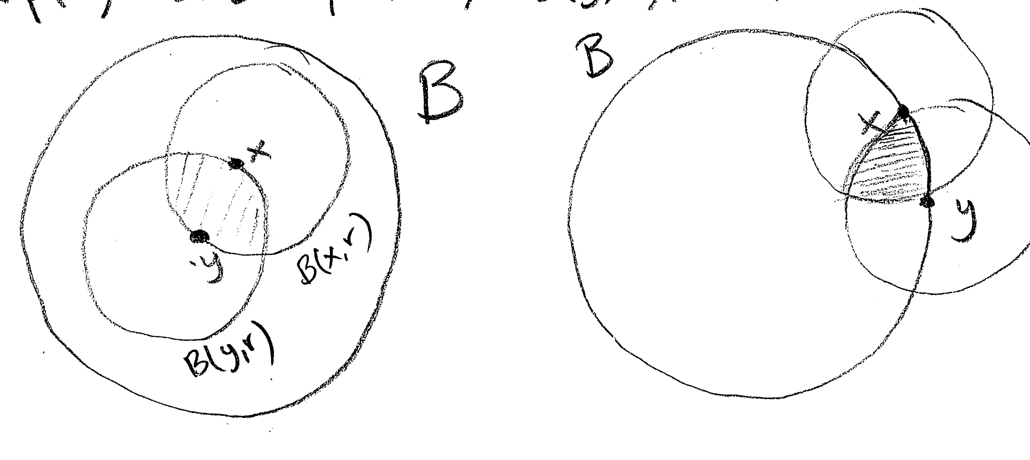
Letting $k \to \infty$, from $(***)$:

$$(A) \quad \boxed{|u(x) - v_r(x)| \leq C(n,p,\gamma)\,\alpha\, r^\gamma}, \quad x \text{ Lebesgue point of } u$$

Now, let $x, y \in B$, $r = |x-y|$

$\Rightarrow \exists \beta(n)$ s.t. $|B(x,r) \cap B(y,r)| = \beta(n) r^n$.

$\Rightarrow \exists \beta'(n) < \beta(n)$ s.t. $|B(x,r) \cap B(y,r) \cap B| \geq \beta'(n) r^n$



$\Rightarrow$

$$|v_r(x)-v_r(y)| \le |v_r(x)-u(z)+u(z)-v_r(y)|$$

$$|v_r(x)-v_r(y)|^p \le 2^{p-1}\left(|v_r(x)-u(z)|^p+|u(z)-v_r(y)|^p\right)$$

$$\Rightarrow \underbrace{\int_{B\cap B(x,r)\cap B(y,r)} |v_r(x)-v_r(y)|^p\,dz}_{\geq\ |v_r(x)-v_r(y)|^p \beta'(n) r^n} \le 2^{p-1}\left(\int_{B\cap B(x,r)\cap B(y,r)} |v_r(x)-u(z)|^p\,dz+\int_{B\cap B(x,r)\cap B(y,r)} |u(z)-v_r(y)|^p\,dz\right)$$

$\therefore$

$$\beta'(n)r^n\,|v_r(x)-v_r(y)|^p \le 2^{p-1}\left(\int_{B\cap B(x,r)} |u(z)-(u)_{r,x}|^p\,dz+\int_{B\cap B(y,r)} |u(z)-(u)_{r,y}|^p\,dz\right)$$

Again, by hypothesis:

$$(B)\quad \boxed{|v_r(x)-v_r(y)| \le C(n,p)\alpha r^\gamma = C(n,p)\alpha|x-y|^\gamma}$$

Thus, if x,y are Lebesgue points of $\chi_B u$:

$$|u(x)-u(y)| \le |u(x)-v_r(x)+v_r(x)-v_r(y)+v_r(y)-u(y)|$$

$$\le |u(x)-v_r(x)|+|v_r(x)-v_r(y)|+|v_r(y)-u(y)|$$

$$\le C(n,p,\gamma)\alpha r^\gamma + C(n,p)\alpha r^\gamma + C(n,p,\gamma)\alpha r^\gamma, \text{ from } A+B.$$

$$= C(n,p,\gamma)\alpha|x-y|^\gamma;\quad r=|x-y|.$$

$$\Rightarrow \boxed{|u(x)-u(y)| \le 3C(n,p,\gamma)\alpha|x-y|^\gamma,\ \forall x,y \text{ Lebesgue points of } \chi_B u}$$

Note the (E) implies that for every $r>0$:

$\{v_{r_k}\}_{k=1}^{\infty}$ is uniformly Cauchy on B

$\{v_{r_k}\}$ continuous functions (and bounded)

$\Rightarrow \exists\ \bar{u} : B \to R$ s.t:

$v_{r_k} \to \bar{u}$ uniformly on B.

$\Rightarrow v_{r_k}(x) \to \bar{u}(x)$, $x \in B$.

Since $v_{r_k}(x) \to u(x)$, $\mathcal{L}^n$ - a.e. x

$\Rightarrow \bar{u}(x) = u(x)$, $\mathcal{L}^n$-a.e. x

$\Rightarrow |\bar{u}(x) - \bar{u}(y)| \leq C(n, p, \gamma)\, \alpha\, |x-y|^{\gamma}$, x, y Lebesgue points of u

Since $\bar{u}$ is continuous we conclude:

$$|\bar{u}(x) - \bar{u}(y)| \leq c(n, p, \gamma)\, \alpha\, |x-y|^{\gamma} \quad \forall x, y \in B.$$