Lection 7

Lower dimensional densities of a Radon measure.

μ Radon measure on $\mathbb{R}^n$.

$s \in (0, n]$.

The upper s-dimensional density $\Theta_s^*(\mu) : \mathbb{R}^n \to [0,\infty]$ is:

$$\Theta_s^*(\mu)(x) = \limsup_{r \to 0^+} \frac{\mu(\bar{B}(x,r))}{\omega_s r^s}, \quad x \in \mathbb{R}^n.$$

$\Theta_s^*(\mu)$ is a Borel function. If this limit exits, then we call that number the "s-dimensional density of μ at x",

$$\Theta_s(\mu)(x) = \lim_{r \to 0^+} \frac{\mu(\bar{B}(x,r))}{\omega_s r^s}.$$

Remark: Note that when $s=n$,

$$\Theta_n^*(\mu)(x) = \limsup_{r \to 0^+} \frac{\mu(\bar{B}(x,r))}{\omega_n r^n} = \lim_{r \to 0^+} \frac{\mu(\bar{B}(x,r))}{\mathcal{L}^n(\bar{B}(x,r))},$$

which means that we can study this case by the Lebesgue-Besicovitch differentiation theorem ($\mathcal{L}^n$ is a Radon measure). But $s<n$ is more complex.

We have:

Theorem: μ a Radon measure on $\mathbb{R}^n$, M Borel set, $s \in (0,n)$, then:

$$\Theta_s^*(\mu) \geq 1 \text{ on } M \Rightarrow \mu(M) \geq \mathcal{H}^s(M) \quad (A)$$

$$\Theta_s^*(\mu) \leq 1 \text{ on } M \Rightarrow \mu(M) \leq 2^s \mathcal{H}^s(M) \quad (B)$$

Proof:

For (A), WLOG, $M \subset B_R$, $R>0$

Claim: $\Theta_s^*(\mu) \geq 1$ on $M \Rightarrow \mathcal{H}^s(M) < \infty$.

Fix $\delta > 0$. Define:

$$\mathcal{F} = \{ \bar{B}(x,r) : x \in M,\ 2r < \delta,\ \mu(\bar{B}(x,r)) \geq (1-\delta)\omega_s r^s \}$$

We can apply Besicovitch covering theorem to $\mathcal{F}$ to obtain $\{\mathcal{F}_i\}_{i=1}^{c(n)}$ subfamilies of $\mathcal{F}$ and:

$$\mathcal{H}_\delta^s(M) \leq \sum_{i=1}^{c(n)} \sum_{\bar{B} \in \mathcal{F}_i} \omega_s \left(\frac{\operatorname{diam}(\bar{B})}{2} \right)^s$$

$$\leq \sum_{i=1}^{c(n)} \sum_{\bar{B} \in \mathcal{F}_i} \frac{1}{(1-\delta)} \mu(\bar{B}(x,r))$$

$$\leq \frac{c(n)}{(1-\delta)} \max_{1 \leq i \leq c(n)} \mu\left(\bigcup_{\bar{B} \in \mathcal{F}_i} \bar{B} \right) \leq \frac{c(n)}{1-\delta} \mu(B_{R+\delta})$$

Letting $\delta \to 0^+ \Rightarrow \mathcal{H}^s(M) < \infty$.

Since $\mathcal{H}^s(M) < \infty \Rightarrow \mathcal{H}^s \llcorner M$ is Radon measure and thus we can apply Vitali's property (the Corollary to Bisicovitch covering theorem) to $\mathcal{H}^s \llcorner M$.

Let A be an open set with $M \subset A$

Define:

$$\mathcal{F}' = \{\bar{B}(x,r) : x \in M,\ 2r < \delta,\ \bar{B}(x,r) \subset A,\ \mu(\bar{B}(x,r)) \geq (1-\delta)\omega_s r^s\}.$$

This covering of M satisfies the hypothesis of Vitali's property (Since $\forall x \in M$, $\exists \{r_k\}$, $r_k \to 0$ with

$$\lim_{k\to\infty} \frac{\mu(\bar{B}(x,r_k))}{\omega_s r_k^s} \geq 1 \Rightarrow \mu(\bar{B}(x,r_k)) \geq (1-\delta)\omega_s r_k^s \text{ for}$$

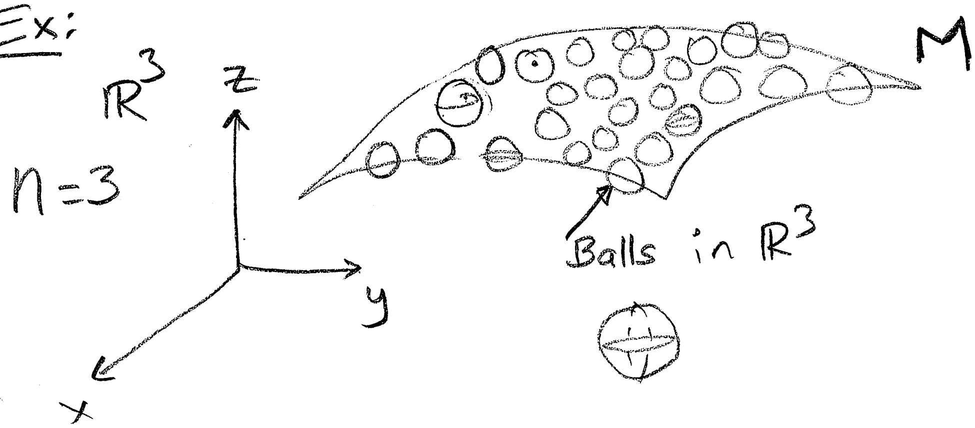
k large enough $\Rightarrow \inf\{\text{diam } \bar{B},\ \bar{B} \text{ has center at } x\} = 0$).

Hence $\exists\ \mathcal{G} \subset \mathcal{F}'$, disjoint family such that:

$$\mathcal{H}^s\left(M \setminus \left(\bigcup_{i=1}^{\infty} \bar{B}_i\right)\right) = 0, \qquad \mathcal{G} = \bigcup_{i=1}^{\infty} \bar{B}_i$$

Let $N = M \setminus \bigcup_{i=1}^{\infty} \bar{B}_i$

<u>Ex:</u>



$s = 2$

$$\limsup_{r\to 0} \frac{\mu(\bar{B}(x,r))}{\omega_2 r^2} \geq 1$$

$\forall x \in M$.

Since $\mathcal{H}^s(N) = 0 \Rightarrow \mathcal{H}^s_\delta(N) = 0$. Hence

$$\mathcal{H}^s_\delta(M) = \sum_{i=1}^{\infty} \omega_s \left(\frac{\text{diam } \bar{B}_i}{2}\right)^s \leq \frac{1}{1-\delta} \sum_{i=1}^{\infty} \mu(\bar{B}_i) \leq \frac{1}{1-\delta}\, \mu\left(\bigcup_{i=1}^{\infty} \bar{B}_i\right)$$

$$\leq \frac{1}{1-\delta}\, \mu(A)$$

$\Rightarrow \mathcal{H}^s_\delta(M) \le \frac{1}{1-\delta}\mu(A)$, $\forall A$ open $M \subset A$

Since μ is Radon $\Rightarrow \mu(M) = \inf\{\mu(A),\ M \subset A,\ A \text{ open}\}$

Thus, letting $\delta \to 0^+$ gives:

$$\boxed{\mathcal{H}^s(M) \le \mu(E)}$$

Part (B) can be proved with similar proof, covering M with closed balls. ∎

Corollary: $s \in (0,n)$, $M \subset \mathbb{R}^n$ Borel, $\mathcal{H}^s(M \cap K) < \infty$, $\forall K \subset \mathbb{R}^n$, K compact. Then:

$$\lim_{r \to 0^+} \frac{\mathcal{H}^s(M \cap B(x,r))}{\omega_s r^s} = 0 \ , \quad \mathcal{H}^s\text{-a.e. } x \in \mathbb{R}^n \setminus M.$$

Proof:

M

$n=3$
$s=2$

WLOG, $\mathcal{H}^s(M) < \infty$. Fix $\delta > 0$

$F_\delta := \{x \in \mathbb{R}^n \setminus M : \Theta^{*}_s(\mu)(x) \ge \delta\}$, $\mu = \mathcal{H}^s \llcorner M$ (Radon)

Previous theorem $\Rightarrow \underbrace{(\mathcal{H}^s \llcorner M)(F_\delta)}_{= \mathcal{H}^s(M \cap F_\delta) = 0} \ge \delta\, \mathcal{H}^s(F_\delta)$

$\Rightarrow \mathcal{H}^s(F_\delta) = 0$. Since $\{x \in \mathbb{R}^n \setminus M : \Theta^{*}_s(\mu)(x) > 0\} = \bigcup_{k=1}^{\infty} F_{1/k}$, the result follows.

Lipschitz functions

They play a special role in Geometric Measure Theory because:

(a) Various metric properties which are characteristic of C^1-functions are also satisfied by Lipschitz functions.

(b) The Lipchitz condition $|f(x)-f(y)| \le Lip(f)|x-y|$ is stable under plain pointwise convergence and can be formulated in terms of set inclusions only. These two features are very compatible with measure-theoretic arguments.

Recall: $f: E \subset \mathbb{R}^n \to \mathbb{R}^m$

$$Lip(f;E) = \sup\left\{\frac{|f(x)-f(y)|}{|x-y|} \; ; \; x,y \in E, \; x \neq y\right\} < \infty$$

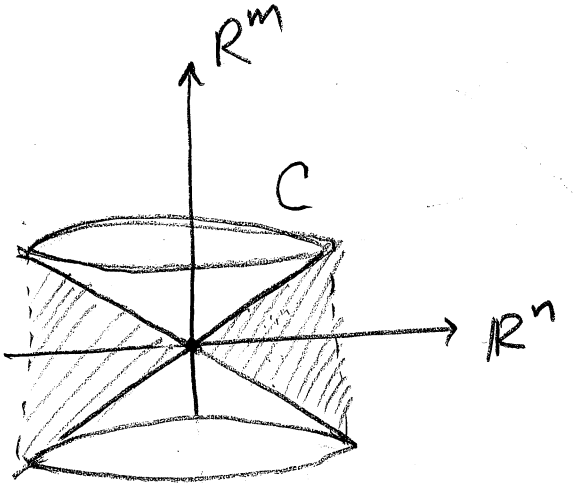
$$Lip(f;\mathbb{R}^n) = Lip(f).$$

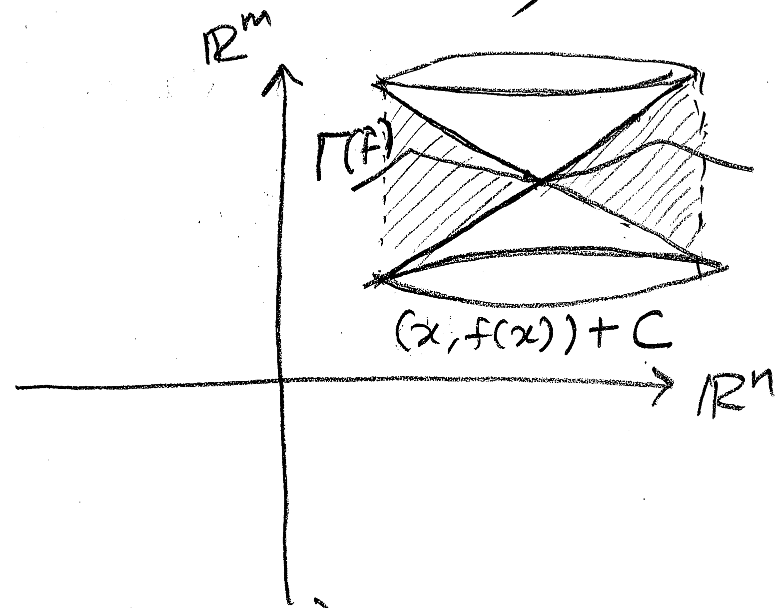
Geometric interpretation: For every $x \in \mathbb{R}^n$, the graph of f, $\Gamma(f) = \{(x,f(x)), x \in \mathbb{R}^n\}$ is cointained in the cone of vertex $(x, f(x))$ and opening $Lip(f)$. That is;

$$\Gamma(f) \subset \bigcap_{x \in \mathbb{R}^n} (x, f(x)) + \{(z,w) \in \mathbb{R}^n \times \mathbb{R}^m, \; |w| \le Lip(f)|z|\}$$

where the cone is

$$C = \{(z,w) \in \mathbb{R}^n \times \mathbb{R}^m : |w| \leq \mathrm{Lip}(f)|z|\}$$





<u>Theorem</u>: (Kirszbraun's theorem):
$E \subset \mathbb{R}^n$, $f : E \to \mathbb{R}^m$ Lipschitz. Then $\exists g : \mathbb{R}^n \to \mathbb{R}^m$,
s.t. $g|_E = f$, $\mathrm{Lip}(g) = \mathrm{Lip}(f; E)$

We can try to prove this theorem as follows: Define g on $x \in \mathbb{R}^n$ as:

$$(*) \begin{cases} g_1(x) = \inf\limits_{y \in E} \{f(y) + \mathrm{Lip}(f;E)|x-y|\} & \text{(highest)} \\ \text{or} \\ g_2(x) = \sup\limits_{y \in E} \{f(y) - \mathrm{Lip}(f;E)|x-y|\} & \text{(lowest)} \end{cases}$$

Both definitions work for $\boxed{m=1}$. However, for $m > 1$, and using $(*)$ for each component we get $g : \mathbb{R}^n \to \mathbb{R}^m$ with the non-optimal bound:

$$\mathrm{Lip}(g) \leq \sqrt{m}\, \mathrm{Lip}(f; E)$$

We need another way to prove Kirszbraun's theorem.
We will use the following:

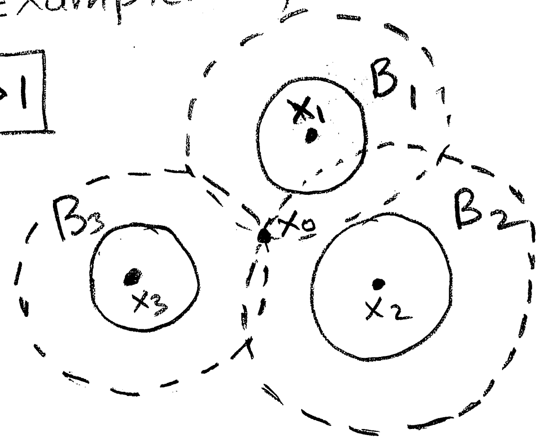
<u>Lemma</u>: Given a finite collection of closed balls $\{\overline{B}(x_k, r_k)\}_{k=1}^{N}$ in $\mathbb{R}^n$, set:

$$C_t = \bigcap_{k=1}^{N} \overline{B}(x_k, t r_k), \quad t \geq 0$$

Let $s = \inf\{t \geq 0 ; C_t \neq 0\}$, then $s < \infty$ and $C_s = \{x_0\}$, x_0 belongs to the convex hull of those x_k such that $|x_0 - x_k| = s r_k$

Examples of this Lemma:

$\boxed{s > 1}$



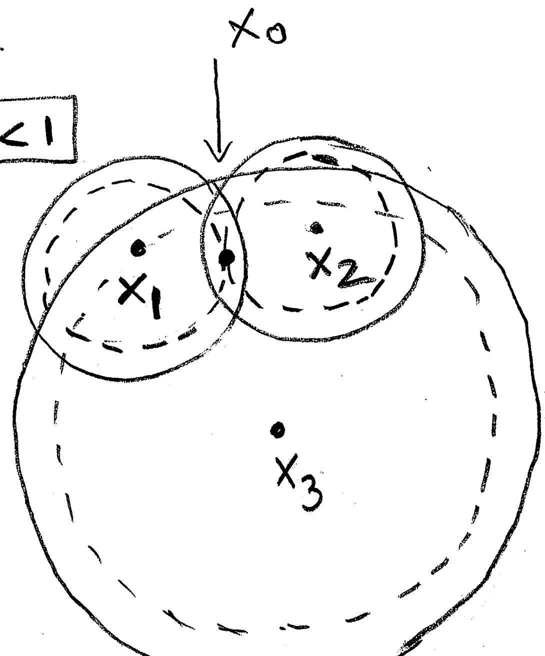
$X_0 \in$ Convel Hull $\{x_1, x_2, x_3\}$

$|x_0 - x_1| = s r_1$

$|x_0 - x_2| = s r_2$

$|x_0 - x_3| = s r_3$

$\boxed{s < 1}$



$X_0 \in$ Convex Hull $\{x_1, x_2\}$

x_3 is not needed

$|x_0 - x_1| = r_1$

$|x_0 - x_2| = r_2$

Proof of Kirsbraun's theorem:

We have:

<u>Claim</u>: If $E \subset \mathbb{R}^n$, $E \neq \mathbb{R}^n$, f Lipschitz, then we can extend f to $E \cup \{y\}$, $y \in \mathbb{R}^n \setminus E$ such that the new function g satisfies:

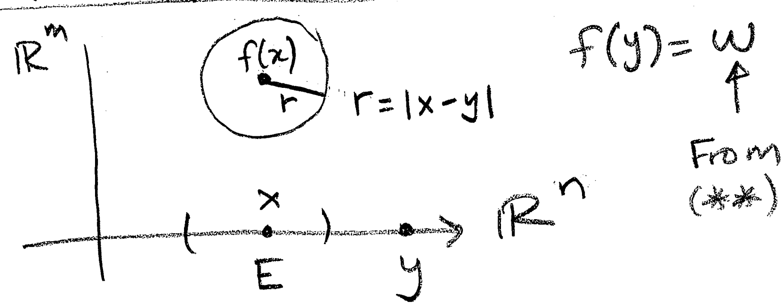
$$\mathrm{Lip}(g; E \cup \{y\}) = \mathrm{Lip}(f; E)$$

WLOG (up to replacing f with λf) we can assume $\mathrm{Lip}(f; E) = 1$.

We need to show $\exists w \in \mathbb{R}^m$ s.t.:
$$|f(x) - w| \leq |x - y|, \quad \forall x \in E \qquad (*)$$

This is equivalent to show:

$$\bigcap_{x \in E} \bar{B}(f(x), |x-y|) \neq \emptyset \qquad (**)$$



If every finite collection has nonempty intersection, then (by compactness) the total intersection $(*)$ will have nonempty intersection. Thus, we take:

$\{x_k\}_{k=1}^N \subset E$, and consider $\{\bar{B}(f(x_k), r_k\}$, $r_k = |x_k - y|$

By previous Lemma $\exists s>0, \exists x_0$ s.t.

$$\bigcap_{k=1}^{N} \bar{B}(f(x_k), s|x_k-y|) = \{x_0\}$$

Let

$$\boxed{z := x_0}$$

Also, up to a permutation of the $\{x_k\}$, and for some $1 \le M \le N$:

$$\{z\} = \bigcap_{k=1}^{N} \bar{B}(f(x_k), s|y-x_k|); \quad |z-f(x_k)| = s|y-x_k|, \quad z = \sum_{k=1}^{M} \lambda_k f(x_k),$$

where $\lambda_k > 0$, $1 \le k \le M$, $1 = \sum_{k=1}^{M} \lambda_k$.

Now,

$$z = \sum_{k=1}^{M} \lambda_k z \Rightarrow \sum_{k=1}^{M} \lambda_k (z - f(x_k)) = 0$$

$$\Rightarrow 0 = 2\left|\sum_{k=1}^{M} \lambda_k (z-f(x_k))\right|^2$$

$$\underset{\text{algebra}}{=} \sum_{k,i=1}^{M} \lambda_k \lambda_i \left(\underbrace{|z-f(x_k)|^2}_{\text{Lemma}} + \underbrace{|z-f(x_i)|^2}_{\text{Lemma}} - \underbrace{|f(x_k)-f(x_i)|^2}_{\text{Lipschitz}}\right)$$

$$\geq \sum_{k,i=1}^{M} \lambda_k \lambda_i \left(s^2|y-x_k|^2 + s^2|y-x_i|^2 - |x_k - x_i|^2\right)$$

$$\underset{\text{algebra}}{=} 2s^2 \left|\sum_{k=1}^{M} \lambda_k (y-x_k)\right|^2 + (s^2-1) \sum_{k,i=1}^{M} \lambda_k \lambda_i |x_k - x_i|^2$$

Thus, either $M=1$ and $s=0$, or $M>1$ and $s \le 1$

$\therefore |z-f(x_k)| \le |y-x_k|, \forall k=1,\dots,N \Rightarrow \bigcap_{k=1}^{N} \bar{B}(f(x_k), |y-x_k|) \neq \emptyset$.

Hence (**) and (*) is true. Claim is proved. ∎

We can now finish the proof of Kirszbraun's theorem: Let $\mathcal{G}$ be the set of pairs (g, F), $E \subset F$, $g: F \to \mathbb{R}^m$, $g = f$ on E and $\mathrm{Lip}(g; F) \leq \mathrm{Lip}(f; E)$. We set;

$$(g_1, F_1) \preceq (g_2, F_2) \text{ if } F_1 \subset F_2,\ g_2 = g_1 \text{ on } F_1.$$

$\preceq$ defines an order $\Rightarrow$ By Zorn's lemma $\exists (g, F)$ maximal and $F = \mathbb{R}^n$ by previous Claim. ∎

<u>Differentiability</u>

Theorem (Rademacher's Theorem):
Let $f: \mathbb{R}^n \to \mathbb{R}$ Lipschitz, then f is differentiable $\mathcal{L}^n$-a.e. (also true for $m > 1$).

<u>Proof</u>;

Please read proof in Lecture #25 and #26 from the notes on my website corresponding to: "Advanced Topics in Analysis".