Lecture 8

Rectifiable sets:

Def: Let $M \subset \mathbb{R}^n$, $\mathcal{H}^k$-measurable.
We say that M is countably $\mathcal{H}^k$-rectifiable if there exists countably many Lipschitz maps $f_i : \mathbb{R}^k \to \mathbb{R}^n$ such that:

$$\mathcal{H}^k\left(M \setminus \bigcup_{i=1}^{\infty} f_i(\mathbb{R}^k)\right) = 0$$

- We say that M is locally $\mathcal{H}^k$-rectifiable if M is countably $\mathcal{H}^k$-rectifiable and, $\mathcal{H}^k(M \cap K) < \infty$, $\forall K \subset \mathbb{R}^n$ compact

- We say that M is $\mathcal{H}^k$-rectifiable if M is countably $\mathcal{H}^k$-rectifiable and $\mathcal{H}^k(M) < \infty$.

It is possible to decompose a countably $\mathcal{H}^k$-rectifiable set by means of (almost flat) regular Lipschitz images, this is the statement of the next Theorem, whose detailed proof is long and can be found in the textbook:

Theorem 1: M countably $\mathcal{H}^k$-rectifiable in $\mathbb{R}^n$. Then $\exists M_0$, $M_0 \subset \mathbb{R}^n$ Borel and $f_i : \mathbb{R}^k \to \mathbb{R}^n$ Lipschitz and $E_i \subset \mathbb{R}^k$ Borel sets such that:

$$\boxed{M = M_0 \cup \left(\bigcup_{i=1}^{\infty} f_i(E_i)\right), \mathcal{H}^k(M_0) = 0} \quad (*)$$

Moreover, for every $t>1$ a decomposition (*) can be found such that: 8.2

- Each (f_i, E_i) defines a regular Lipschitz image, $Lip(f_i) \le t$
- $\frac{1}{t}|x-y| \le |f_i(x) - f_i(y)| \le t|x-y|$
- $\frac{1}{t}|v| \le |\nabla f_i(x)\, v| \le t|v|$
- The Jacobian Jf_i satisfies $\frac{1}{t^k} \le Jf_i(x) \le t^k$

Ideas about proof of Theorem 1:

The notion "regular Lipschitz image" means that

(a) f_i is 1-1 and differentiable on E_i, $Jf(x) > 0$, $\forall x \in E_i$.

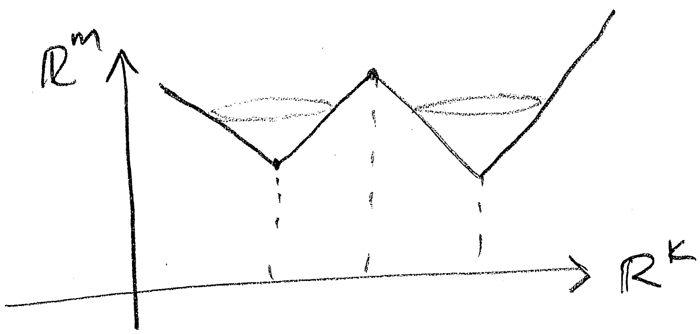
(b) $x \in E_i^{(1)}$, $\forall x \in E_i$; i.e. $\lim_{r\to 0} \frac{|E_i \cap B(x,r)|}{\omega_k r^k} = 1$, $\forall x \in E_i$

(c) Every $x \in E_i$ is a Lebesgue point of ∇f_i.

The main 2 ingredients in the proof of this Theorem 1 are:

(A) The fact that if $f: \mathbb{R}^k \to \mathbb{R}^n$ is Lipschitz then (note that $k \le n$):

$$\mathcal{H}^k(f(E)) = 0, \quad E = \{x \in \mathbb{R}^k : Jf(x)\} = 0$$



(B) The second ingredient is the so called "linearization of a Lipschitz function on the set $\{Jf > 0\}$". The Jacobian is defined as:

$$Jf(x) = \begin{cases} \sqrt{\det(\nabla f(x))^* \nabla f(x))}, & \text{if } f \text{ is differentiable at } x \\ +\infty, & \text{if } f \text{ is not differentiable at } x \end{cases}$$

Note: for $f : \mathbb{R} \to \mathbb{R}$ Lipschitz, $Jf(x) = |f'(x_0)|$

For $f : \mathbb{R}^n \to \mathbb{R}^n$ Lipschitz then

$$Jf(x) = \sqrt{\det(\nabla f(x))^* \det(\nabla f(x))} = \sqrt{\det(\nabla f(x))^2} = |\det \nabla f(x)|$$

Also, recall that, if $f = (f_1, \dots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ then $\nabla f(x)$ is the $n \times n$ matrix:

$$\nabla f(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

If $f : \mathbb{R}^K \to \mathbb{R}^n$, $K \neq n$, and $f = (f_1, \dots, f_n)$ then:

$$\nabla f(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_K} \\ & \vdots & \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix},$$ an $n \times K$ matrix.

This linearization of Lipschitz maps roghly speaking says that $F=\{x: 0<Jf(x)<\infty\}$ can be decomposed in a countable partition such that $\nabla f(x)$ is almost constant on each set in the partition. This is done (Federe's idea) by fixing a set $\{T_i\}_{i=1}^{\infty} \subset \mathbb{R}^n \otimes \mathbb{R}^k$, where:

$$\mathbb{R}^n \otimes \mathbb{R}^k := \{T: \mathbb{R}^k \to \mathbb{R}^n, \text{ linear maps}\},$$

and then choosing $\delta>0$, look at the Borel sets of those $x \in \mathbb{R}^k$ such that $\nabla f(x)$ is δ-close to a fixed T_i. More precisely, if $f: \mathbb{R}^k \to \mathbb{R}^n$ is Lipschitz, with F as above, there exists a partition $\{F_i\}$, F_i Borel f is 1-1 on F_i. Moreover, for every $t>1$, such partition of F can be found with the property that, $\forall i=1,2,\ldots$, $\exists S_i \in GL(k)$ (i.e: $S_i: \mathbb{R}^k \to \mathbb{R}^k$ linear, invertible) such that $f|_{F_i} \circ S_i^{-1}: \mathbb{R}^k \to \mathbb{R}^n$ is almost an isometry; that is, $\forall x,y \in F_i$, $\forall v \in \mathbb{R}^k$:

$$\frac{1}{t}|S_i x - S_i y| \le |f(x)-f(y)| \le t\,|S_i x - S_i y|$$

$$\frac{1}{t}|S_i v| \le |\nabla f(x) v| \le t\,|S_i v|$$

$$\frac{1}{t^n} JS_i \le Jf(x) \le t^n JS_i$$

Going back to the proof of theorem 1, we have M locally $\mathcal{H}^k$- rectifiable with

$$M \subset \left(\bigcup_{i=1}^{\infty} f_i(\mathbb{R}^k) \right) \cup \tilde{M}_0, \quad \mathcal{H}^k(\tilde{M}_0) = 0$$

With Rademacher's theorem, (A) and (B) applied to each i we obtain:

$$M = M_0 \cup \left(\bigcup_{i=1}^{\infty} g_i(G_i) \right), \quad \mathcal{H}^k(M_0) = 0,$$

where each (g_i, G_i) defines a regular Lipschitz image. By Kirszbaum's theorem, we exted each g_i to $f_i : \mathbb{R}^k \to \mathbb{R}^n$ with $\mathrm{Lip}(f_i) \leq t$. One has to check that all the desired properties of f_i are satisfied, using (B). ∎

Approximate tangent spaces to rectifiable sets,

Recall from previous lectures:

- $f: \mathbb{R}^n \to \mathbb{R}^m$

  μ measure on $\mathbb{R}^n$

  $f_{\#}\mu$, the push-forward of μ through f

is a measure on $\mathbb{R}^m$ defined as:

$$f_{\#}\mu(E) = \mu(f^{-1}(E)), \quad E \subset \mathbb{R}^m.$$

- $$\int_{\mathbb{R}^m} u \, d(f_{\#}\mu) = \int_{\mathbb{R}^n} (u \circ f) \, d\mu, \quad \forall u \text{ Borel}$$

- $\Phi_{x,r}: \mathbb{R}^n \to \mathbb{R}^n$, $\Phi_{x,r}(y) = \dfrac{y-x}{r}$

$$(\Phi_{x,r})_{\#}\mu(E) = \mu(x + rE).$$

We have:

<u>Theorem 2</u>: (Existence of approximate tangent spaces). $M \subset \mathbb{R}^n$ locally $\mathcal{H}^k$-rectifiable set, then for $\mathcal{H}^k$-a.e. $x \in M$, $\exists$ unique k-dimensional plane Π_x such that, as $r \to 0^+$,

$$\frac{(\Phi_{x,r})_{\#}(\mathcal{H}^k \llcorner M)}{r^k} = \mathcal{H}^k \llcorner \left(\frac{M-x}{r}\right) \xrightarrow{*} \mathcal{H}^k \llcorner \Pi_x,$$

which is:

$$\lim_{r\to 0^+} \frac{1}{r^k}\int_M \varphi\left(\frac{y-x}{r}\right) d\mathcal{H}^k(y) = \int_{\Pi_x} \varphi \, d\mathcal{H}^k, \quad \forall \varphi \in C_c(\mathbb{R}^n).$$

In particular, $\Theta_k(\mathcal{H}^k \llcorner M) = 1$, $\mathcal{H}^k$-a.e. on M, that is:

$$\lim_{r\to 0^+} \frac{(\mathcal{H}^k \llcorner M)(B(x,r))}{\omega_k r^k}$$

$$= \lim_{r\to 0^+} \frac{\mathcal{H}^k(M \cap B(x,r))}{\omega_k r^k} = 1, \quad \mathcal{H}^k\text{-a.e. } x \in M$$

Notation: $\Pi_x = T_x M$, the approximate tangent space to M at x.

Proof: We apply Theorem 1 to M to obtain the decomposition:

$$M = M_0 \cup \left(\bigcup_{i=1}^{\infty} f_i(E_i)\right).$$

Let $M_i := f_i(E_i)$

For each i, since $Jf_i(x) \neq 0$, $\forall x \in E_i$, $\Pi_x := \nabla f_i(x)(\mathbb{R}^k)$ is a k-dimensional plane. Since $M_i = f_i(E_i)$ is a k-dimensional regular Lipschitz image in $\mathbb{R}^n$ then

$$T_x M_i = \nabla f_i(x)(\mathbb{R}^k)$$ (This will proved next Lec. in Lemma 2).

that is,

$$\lim_{r\to 0^+} \frac{1}{r^k}\int_{M_i} \varphi \circ \Phi_{x,r} \, d\mathcal{H}^k = \int_{\Pi_x = T_x M_i} \varphi \, d\mathcal{H}^k, \quad \forall x \in M_i \quad (1)$$

By Corollary proved in Lecture 7 (page 7.4), which we can apply since $\mathcal{H}^k \llcorner M$ is a Radon measure, we have:

$$\Theta_k(\mathcal{H}^k \llcorner (M \setminus M_i)) = \lim_{r \to 0} \frac{\mathcal{H}^k((M \setminus M_i) \cap B(x,r))}{\omega_k r^k} = 0,$$

for $\mathcal{H}^k$-a.e. $x \in M_i$. In particular, this implies

$$\lim_{r \to 0^+} \frac{1}{r^k} \int_{M \setminus M_i} \varphi \circ \Phi_{x,r} \, d\mathcal{H}^k =$$

$$= \lim_{r \to 0^+} \frac{1}{r^k} \int_{M \setminus M_i} \varphi\left(\frac{y-x}{r}\right) d\mathcal{H}^k \quad , \quad \varphi \in C_c(\mathbb{R}^n), \ \operatorname{supp} \varphi \subset B(0,N)$$

$$= \lim_{r \to 0^+} \frac{1}{r^k} \int_{B(x,rN) \cap (M \setminus M_i)} \varphi\left(\frac{y-x}{r}\right) d\mathcal{H}^k$$

$$\leq C \omega_k \lim_{r \to 0^+} \frac{\mathcal{H}^k(B(x,rN) \cap (M \setminus M_i))}{\omega_k r^k}$$

$$= 0, \quad \text{for } \mathcal{H}^k\text{-a.e. } x \in M_i. \qquad (2)$$

Hence, from (1) and (2):

$$\lim_{r \to 0^+} \frac{1}{r^k} \int_M \varphi\left(\frac{y-x}{r}\right) d\mathcal{H}^k(y) = \lim_{r \to 0^+} \frac{1}{r^k} \int_{M_i} \varphi\left(\frac{y-x}{r}\right) d\mathcal{H}^k(y)$$

$$+ \lim_{r \to 0^+} \frac{1}{r^k} \int_{M \setminus M_i} \varphi\left(\frac{y-x}{r}\right) d\mathcal{H}^k(y)$$

$$= \int_{\Pi_x} \varphi \, d\mathcal{H}^k + 0 = \int_{\Pi_x} \varphi \, d\mathcal{H}^k$$

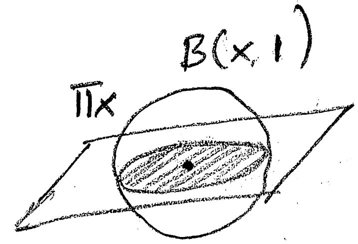
Thus:

Since $\mathcal{H}^k$-a.e. $x \in M$ belongs to some $M_i = f_i(E_i)$ we conclude

$$\lim_{r\to 0^+} \frac{1}{r^k}\int_M \varphi\left(\frac{y-x}{r}\right) d\mathcal{H}^k(y) = \int_{\Pi_x = T_x M_i} \varphi \, d\mathcal{H}^k \quad , \ \mathcal{H}^k\text{-a.e. } x,$$

which is the first part of Theorem 2. Let $x \in M$ which satisfies the above limit. Since $\Pi_x \cap \partial B$ is a $(k-1)$-dimensional sphere, with $\mathcal{H}^k \llcorner \Pi_x(\partial B) = 0$. Thus:

$$\omega_k = \mathcal{H}^k \llcorner \Pi_x(B) \qquad B = B(x,1)$$



$\mathcal{H}^2(\text{Disk}) = \omega_k$

$$= \lim_{r\to 0^+} \frac{\mathcal{H}^k \llcorner M(\Phi_{x,r}^{-1}(B))}{r_k} \quad ; \quad \text{since } \frac{(\Phi_{x,r})_\# \mathcal{H}^k \llcorner M}{r^k} \xrightarrow{*} \mathcal{H}^k \llcorner \Pi_x$$

and $\mathcal{H}^k \llcorner \Pi_x(\partial B) = 0$ (Lecture 4, page 4.10).

$$= \lim_{r\to 0^+} \frac{\mathcal{H}^k(M \cap B(x,r))}{r^k} \quad ; \quad \text{since } \Phi_{x,B}^{-1}(B) = B(x,r)$$

Hence, we obtain

$$\lim_{r\to 0^+} \frac{\mathcal{H}^k(M \cap B(x,r))}{\omega_k r_k} = 1, \text{ for } \mathcal{H}^k\text{-a.e. } x \in M,$$

which is the second part of Theorem 2.