Lecture 9

In the proof of Theorem 2 in Lecture 8, we used the following:

Lemma 2: If $M=f(E)$ is a k-dimensional regular Lipschitz image in $\mathbb{R}^n$ and $z\in E$, then

$$T_x M = \nabla f(z)(\mathbb{R}^k), \quad x=f(z)$$

Proof:

Recall that (f,E) regular Lipschitz image means $E\subset\mathbb{R}^k$, $f:\mathbb{R}^k\to\mathbb{R}^n$ Lipschitz and:

(a) f is 1-1 and differentiable on E, $Jf(x)>0$ $\forall x\in E$.

(b) Every $x\in E$ is a point of density 1 for E; that is:

$$\frac{|E\cap B(x,r)|}{|B(x,r)|}\to 1, \quad \text{as } r\to 0.$$

(c) Every $x\in E$ is a Lebesgue point of ∇f; that is,

$$\lim_{r\to 0^+}\frac{1}{r^k}\int_{B(x,r)}|Jf(z)-Jf(x)|\,dz=0.$$

Let $\varphi\in C_c(\mathbb{R}^n)$. We need to show that

$$\frac{1}{r^k}\int_{\mathbb{R}^n}\varphi\, d(\Phi_{x,r})_\#(\mathcal{H}^k\llcorner M) = \int_{\nabla f(z)(\mathbb{R}^k)}\varphi\, d\mathcal{H}^k \qquad (A)$$

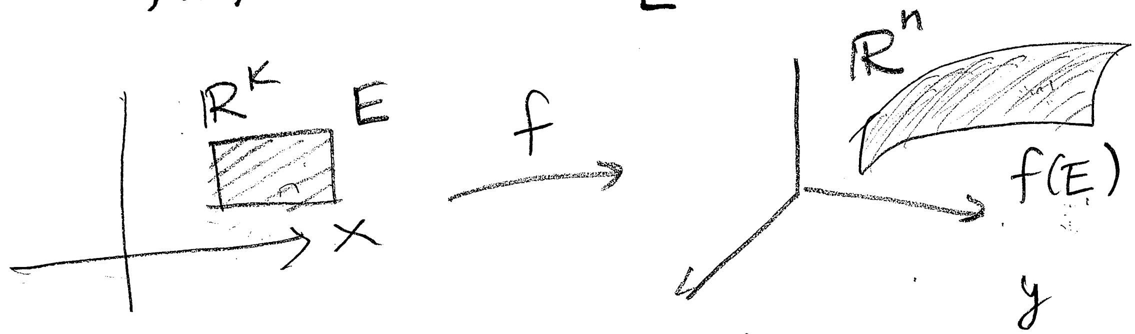
That is, we need to show the weak convergence of measures:

$$\frac{1}{r^K}\left(\Phi_{x,r}\right)_{\#}(\mathcal{H}^K \llcorner M) \xrightarrow{*} \mathcal{H}^K \llcorner \nabla f(z)(\mathbb{R}^K)$$

We need the <u>Area formula</u>, that we will study with more detail later. This formula says:

<u>Area formula</u>: Let $f: \mathbb{R}^K \to \mathbb{R}^n$ Lipschitz, $E \subset \mathbb{R}^K$ be $\mathcal{L}^K$-measurable, f is 1-1 on E, $K \leq n$. Let $g: \mathbb{R}^n \to [-\infty, \infty]$ Borel, $g \in L^1(\mathbb{R}^n, \mathcal{H}^K \llcorner f(E))$. Then

$$\int_{f(E)} g(y)\, d\mathcal{H}^K(y) = \int_E g(f(x))\, Jf(x)\, \underbrace{d\mathcal{L}^K(x)}_{\text{or just } dx}$$



Going back to the proof of (A):

$$\frac{1}{r^K}\int_{\mathbb{R}^n} \varphi \, d(\Phi_{x,r})_{\#}(\mathcal{H}^K \llcorner M) = \frac{1}{r^K}\int_M \varphi \circ \Phi_{x,r}\, d\mathcal{H}^K(y)$$

$$= \frac{1}{r^K}\int_M \varphi\left(\frac{y-x}{r}\right) d\mathcal{H}^K(y)$$

$$= \frac{1}{r^K}\int_E \varphi\left(\frac{f(w)-f(z)}{r}\right) Jf(w)\, dw \; ;$$ since $M = f(E)$ and by area formula

Hence:

$$\frac{1}{r^k}\int_{\mathbb{R}^n} \varphi \, d(\Phi_{x,r})_\#(\mathcal{H}^k \llcorner M)$$

$$= \frac{1}{r^k}\int_E \varphi\left(\frac{f(w)-f(z)}{r}\right) Jf(w)\, dw$$

$$= \int_{\mathbb{R}^k} \chi_E(z+r\alpha)\,\varphi\left(\frac{f(z+r\alpha)-f(z)}{r}\right) Jf(z+r\alpha)\, d\alpha ,$$

where we have applied again the area formula with the transformation:

$$w = z + r\alpha$$

or

$$\alpha = \frac{w-z}{r} , \qquad \alpha, w \in \mathbb{R}^k \quad dw = r^k d\alpha$$

Define:

$$U_r(\alpha) = \chi_E(z+r\alpha)\,\varphi\left(\frac{f(z+r\alpha)-f(z)}{r}\right) Jf(z+r\alpha) ,$$

and for simplicity, let us use again w instead of α, and dw instead of $d\alpha$.

Then:

$$\frac{1}{r^k}\int_{\mathbb{R}^n} \varphi \, d(\Phi_{x,r})_\#(\mathcal{H}^k \llcorner M) = \int_{\mathbb{R}^k} U_r(w)\, dw$$

Since f is differentiable at z, for fixed w, as $r \to 0$: $\frac{f(z+rw)-f(z)}{r} \to \nabla f(z)\, w$. Also, since z is a Lebesgue point of χ_E and Jf (by properties (b) and (c)), then for fixed w, as $r \to 0$ we have:

$$\chi_E(z+rw) Jf(z+rw) \xrightarrow[r\to 0]{} \chi_E(z) Jf(z) = Jf(z)$$

Hence:

$$u_r(w) \longrightarrow \varphi(\nabla f(z) w) Jf(z), \quad \text{as } r \to 0.$$

Moreover, $\|u_r\|_{L^\infty(\mathbb{R}^k)} \leq \sup_{\mathbb{R}^n} |\varphi| \, \mathrm{Lip}(f)^k$ and $\operatorname{spt} u_r \subset B(0,L)$, for some L and r small enough. Hence, we can apply the Lebesgue dominated convergence theorem to obtain:

$$\lim_{r\to 0} \frac{1}{r^k} \int_{\mathbb{R}^n} \varphi \, d(\Phi_{x,r})_\#(\mathcal{H}^k \llcorner M) = \lim_{r\to 0} \int_{\mathbb{R}^k} u_r(w)\, dw$$

$$= \int_{\mathbb{R}^k} \lim_{r \to 0} u_r(w)\, dw$$

$$= \int_{\mathbb{R}^k} \varphi(\nabla f(z) w) Jf(z)\, dw$$

$$= \int_{\nabla f(z)(\mathbb{R}^k)} \varphi \, d\mathcal{H}^k \quad ; \text{ by the Area formula for linear maps.}$$

With $T_x M := \nabla f(z)(\mathbb{R}^k)$, we conclude:

$$\frac{1}{r^k}(\Phi_{x,r})_\#(\mathcal{H}^k \llcorner M) \xrightarrow{*} \mathcal{H}^k \llcorner T_x M. \quad \blacksquare$$

With Lemma 2, we now review again the proof of Theorem 2, which says that:

If M is countably $\mathcal{H}^k$-rectifiable, then for $\mathcal{H}^k$-a.e. $x \in M$, $\exists\, \pi_x$, (a unique k-dimensional plane Π_x) such that:

$$\frac{1}{r^k}(\Phi_{x,r})_\# (\mathcal{H}^k \llcorner M) = \mathcal{H}^k \llcorner \left(\frac{M-x}{r}\right) \xrightarrow{*} \mathcal{H}^k \llcorner \Pi_x . \quad (B)$$

Indeed, we have (by definition of $\mathcal{H}^k$-rectifiability):

$$M \subset \tilde{M}_0 \cup \left(\bigcup_{i=1}^{\infty} f_i(\mathbb{R}^k)\right), \quad f_i : \mathbb{R}^k \to \mathbb{R}^n$$

Lipschitz

$\mathcal{H}^k(\tilde{M}_0) = 0$.

By Lipschitz linearization (Theorem 1), we can decompose M as:

$$M = M_0 \cup \left(\bigcup_{i=1}^{\infty} g_i(E_i)\right), \quad \mathcal{H}^k(M_0) = 0$$

such that $g_i : \mathbb{R}^k \to \mathbb{R}^n$ are Lipschitz and (g_i, E_i) is a regular Lipschitz image. Hence, for $\mathcal{H}^k$-a.e. $x \in M \Rightarrow x \in M_i := g_i(E_i)$ for some E_i. Thus

$$\frac{1}{r^k}\int_{\mathbb{R}^n} \varphi \, d(\Phi_{x,r})_\# (\mathcal{H}^k \llcorner M) = \frac{1}{r^k}\int_M \varphi\left(\frac{y-x}{r}\right) d\mathcal{H}^k(y)$$

$$= \frac{1}{r^k}\int_{M_i} \varphi\left(\frac{y-x}{r}\right) d\mathcal{H}^k(y) + \frac{1}{r^k}\int_{M \setminus M_i} \varphi\left(\frac{y-x}{r}\right) d\mathcal{H}^k(y)$$

Thus,

$$\lim_{r\to 0^+} \frac{1}{r^k}\int_{\mathbb{R}^n} \varphi \, d(\Phi_{x,r})_\#(\mathcal{H}^k \llcorner M)$$

$$= \lim_{r\to 0^+} \frac{1}{r^k}\int_{M_i} \varphi\left(\frac{y-x}{r}\right) d\mathcal{H}^k(y) + \lim_{r\to 0^+} \frac{1}{r^k}\int_{M\setminus M_i} \varphi\left(\frac{y-x}{r}\right) d\mathcal{H}^k(y)$$

$$= \int_{\nabla g_i(z)(\mathbb{R}^k)} \varphi(y)\, d\mathcal{H}^k(y) + \lim_{r\to 0^+} \frac{1}{r^k}\int_{M\setminus M_i} \varphi\left(\frac{y-x}{r}\right) d\mathcal{H}^k(y);$$ by Lemma 2 and $x = g_i(z)$

$$= \int_{\nabla g_i(z)(\mathbb{R}^k)} \varphi \, d\mathcal{H}^k(y) + 0,$$ since, for $\mathcal{H}^k$-a.e. $x \in M_i$:

$$\frac{1}{r^k}\int_{M\setminus M_i} \varphi\left(\frac{y-x}{r}\right) d\mathcal{H}^k(y) = \frac{1}{r^k}\int_{B(x,rR)\cap M\setminus M_i} \varphi\left(\frac{y-x}{r}\right) d\mathcal{H}^k(y)$$, if $\operatorname{spt}\varphi$ is contained in the ball $B(0,R)$

$$\le \sup|\varphi|\, \omega_k R^k \, \frac{|B(x,rR)\cap(M\setminus M_i)|}{\omega_k (rR)^k}$$

$\left|\frac{y-x}{r}\right| \le R \Rightarrow |y-x| \le rR \Rightarrow y \in B(x,rR)$

$\longrightarrow 0$ as $r \to 0$; by Lecture 7, Page 7.4

With

$$\boxed{\Pi_x := \nabla g_i(z)(\mathbb{R}^k)},$$

the desired weak convergence (B) follows. Recall also, the second part of Theorem 2, which says that (See Lecture 8, Page 8.9) if M is countably $\mathcal{H}^k$-rectifiable then:

$$\boxed{\lim_{r\to 0^+} \frac{\mathcal{H}^k(M\cap B(x,r))}{\omega_k r^k} = 1, \text{ for } \mathcal{H}^k\text{-a.e. } x\in M}$$

Remark: Note that in Theorem 2, the uniqueness of the tangent space Π_x follows since, in $M = M_0 \cup \left(\bigcup_{i=1}^{\infty} g_i(E_i)\right)$, the sets $\{g_i(E_i)\}$ are a "partition", not a "covering". So when we say that $\mathcal{H}^k$-a.e. $x \in M$ belongs to some $M_i := g_i(E_i)$, there is not ambiguity in having two such M_i.

However, given any two countably $\mathcal{H}^k$-rectifiable sets M_1 and M_2, they could intersect, and in this case we have the following

Proposition (Locality of approximate tangent spaces).

If M_1 and M_2 are locally $\mathcal{H}^k$-rectifiable sets in $\mathbb{R}^n$, then:

$$T_x M_1 = T_x M_2 \quad , \text{ for } \mathcal{H}^k\text{- a.e. } x \in M_1 \cap M_2$$

Proof: Let $\varphi \in C_c(\mathbb{R}^n)$, $\operatorname{spt} \varphi \subset B(0,R)$, $R > 0$.

Then, if $x \in M_1 \cap M_2$

$$\left| \frac{1}{r^k} \int_{M_1} \varphi\left(\frac{y-x}{r}\right) d\mathcal{H}^k - \frac{1}{r^k} \int_{M_2} \varphi\left(\frac{y-x}{r}\right) d\mathcal{H}^k \right|$$

$$= \frac{1}{r^k} \left| \int_{M_1 \Delta M_2} \varphi\left(\frac{y-x}{r}\right) d\mathcal{H}^k \right| = \frac{1}{r^k} \left| \int_{B(x,rR) \cap (M_1 \Delta M_2)} \varphi\left(\frac{y-x}{r}\right) d\mathcal{H}^k \right|$$

$$\leq \frac{1}{r^k} \sup|\varphi|\, \mathcal{H}^k\left(B(x,rR)\cap(M_1 \Delta M_2)\right)$$

$$= \omega_k R^k \sup|\varphi| \frac{\mathcal{H}^k\left(B(x,rR)\cap(M_1\Delta M_2)\right)}{(rR)^k}$$

(for $\mathcal{H}^k$-a.e. $x \in M_1 \cap M_2$)

$\longrightarrow 0$, as $r \to 0$, by Lecture 7, Page 7.4,

since that useful corollary says that:

$$\lim_{r\to 0^+} \frac{\mathcal{H}^k((M_1 \Delta M_2)\cap B(x,r))}{\omega_k r^k} = 0, \text{ for } \mathcal{H}^k\text{-a.e. } x \in M_1 \cap M_2$$

∎

Now, we want to prove the converse of Theorem 2, but first we show a criterion for Rectifiability. We have.

<u>Rectifiability criterion</u>. Let $M \subset \mathbb{R}^n$ compact, Π is a k-dimensional plane in $\mathbb{R}^n$, and $\exists\, \delta > 0$, $\exists\, t > 0$ such that:

$$M \cap B(x,\delta) \subset x + K(\Pi, t), \quad \forall x \in M,$$

where $K(\Pi,t) = \{y \in \mathbb{R}^n ;\ |y| < \sqrt{1+t^2}\, |P_\Pi y|$,

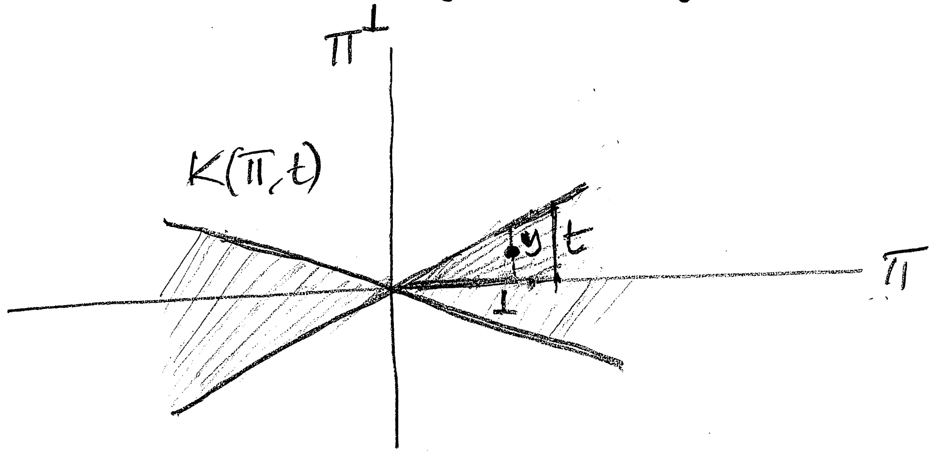
then M is $\mathcal{H}^k$-rectifiable, since there exists finitely many Lipschitz maps $f_i : \mathbb{R}^k \to \mathbb{R}^n$ and compact sets $F_i \subset \mathbb{R}^k$ such that $M = \bigcup_{i=1}^{N} f_i(F_i)$.

<u>Notation</u>: Given a k-dimensional plane π in $\mathbb{R}^n$, let

$$P_\pi : \mathbb{R}^n \to \mathbb{R}^n, \quad P_\pi^\perp : \mathbb{R}^n \to \mathbb{R}^n$$

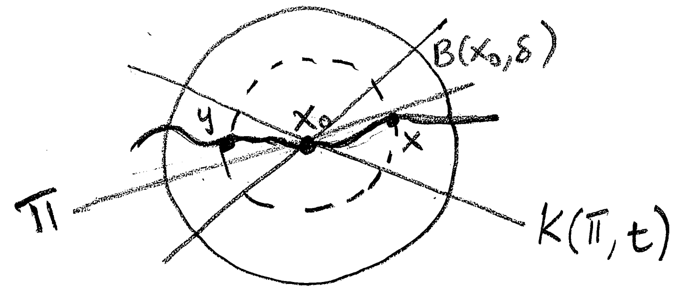
be the orthogonal projections of $\mathbb{R}^n$ onto (respectively) π and $\pi^\perp$ (thus $P_{\pi^\perp} = P_\pi^\perp$), and define the cones $K(\pi,t)$, $t>0$ as

$$K(\pi,t) = \{y \in \mathbb{R}^n : |P_\pi^\perp y| < t\,|P_\pi y|\}$$
$$= \{y \in \mathbb{R}^n : |y| < \sqrt{1+t^2}\,|P_\pi y|\}.$$



<u>Proof of the Rectifiability criterion</u>:

Let $x_0 \in M$ and $x, y \in \overline{B}(x_0, \frac{\delta}{2}) \cap M$



$x, y \in \overline{B}(x_0, \frac{\delta}{2}) \cap M$

$\Rightarrow y \in B(x,\delta) \cap M$

(since $|y-x| \le |y-x_0| + |x_0-x|$
$\le \frac{\delta}{2} + \frac{\delta}{2} = \delta$)

Since $y \in B(x,\delta) \cap M \subset x + K(\pi,t)$

$\Rightarrow |P_\pi^\perp(y-x)| \le t\,|P_\pi(y-x)|$.

Hence,

$$|P_\Pi^\perp(y-x)| \le t\,|P_\Pi(y-x)|, \quad \forall x,y \in \bar{B}(x_0,\tfrac{\delta}{2})\cap M$$

Note that $P_\Pi x = P_\Pi y \Rightarrow P_\Pi^\perp y = P_\Pi^\perp x$

and hence $x=y$. Thus,

$$P_\Pi : \bar{B}(x_0,\tfrac{\delta}{2})\cap M \longrightarrow G_{x_0} = P_\Pi\left(\bar{B}(x_0,\tfrac{\delta}{2})\cap M\right)$$

is 1-1 and on-to. Note, $G_{x_0} \subset \Pi$ compact.

Define now:

$$g_{x_0} : G_{x_0} \longrightarrow \mathbb{R}^n, \quad g_{x_0}(G_{x_0}) = \bar{B}(x_0,\tfrac{\delta}{2})\cap M$$

as $P_\Pi(g_{x_0}(z)) = z$

Then, g_{x_0} is Lipschitz on G_{x_0} because:

$z,w \in G_0 \Rightarrow g_{x_0}(z)$ and $g_{x_0}(w)$ belong to $\bar{B}(x_0,\frac{\delta}{2})\cap M$

$\Rightarrow g_{x_0}(z), g_{x_0}(w)$ are inside $K(\Pi,t)+x_0$

$$\Rightarrow |P_\Pi^\perp(g_{x_0}(z)) - P_\Pi^\perp(g_{x_0}(w))|^2 < t^2 |P_\Pi(g_{x_0}(z)) - P_\Pi(g_{x_0}(w))|^2$$

$$\Rightarrow |P_\Pi(g_{x_0}(z)) - P_\Pi(g_{x_0}(w))|^2 + |P_\Pi^\perp(g_{x_0}(z)) - P_\Pi^\perp(g_{x_0}(w))|^2 < (1+t^2)|\underbrace{P_\Pi(g_{x_0}(z))}_{=z} - \underbrace{P_\Pi(g_{x_0}(w))}_{=w}|^2$$

$$\boxed{\begin{array}{c} |g_{x_0}(z) - g_{x_0}(w)| < \sqrt{1+t^2}\,|z-w| \\ \forall x,z \in \bar{B}(x_0,\tfrac{\delta}{2})\cap M \end{array}}$$

We can cover M with the open balls:

$$\{B(x, \tfrac{\delta}{2}) : x \in M\}$$

By compactness of M, we find $\{x_i\}_{i=1}^{N} \subset M$ with:

$$M = \bigcup_{i=1}^{N} (M \cap \bar{B}(x_i, \tfrac{\delta}{2})) = \bigcup_{i=1}^{N} g_i(G_i),$$

where $g_i : G_i \to \mathbb{R}^n$ are Lipschitz maps and $G_i \subset \Pi$ are compact sets. If $P \in O(k, n)$ is such that:

$$P(\mathbb{R}^k) = \Pi,$$

we then extend g_i to Π using McShane's lemma, which extends Lipschitz functions using $(*)$ in Page 7.6, Lecture 7.

We finally define:

$$f_i = g_i \circ P, \quad F_i = P^{-1}(G_i),$$

and we conclude:

$$M = \bigcup_{i=1}^{N} f_i(F_i). \blacksquare$$