

# MORREY SPACES AND GENERALIZED CHEEGER SETS

QINFENG LI AND MONICA TORRES

ABSTRACT. We maximize the functional

$$\frac{\int_E h(x) dx}{P(E)},$$

where  $E \subset \bar{\Omega}$  is a set of finite perimeter,  $\Omega$  is an open bounded set with Lipschitz boundary and  $h$  is nonnegative. Solutions to this problem are called *generalized Cheeger sets in  $\Omega$* . We show that the Morrey spaces  $L^{1,\lambda}(\Omega)$ ,  $\lambda \geq n-1$ , are natural spaces for  $h$  to study this problem. We prove that if  $h \in L^{1,\lambda}(\Omega)$ ,  $\lambda > n-1$ , then generalized Cheeger sets exist. We also study the embedding of Morrey spaces into  $L^p$  spaces. We show that, for any  $0 < \lambda < n$ , the Morrey space  $L^{1,\lambda}(\Omega)$  is not contained in any  $L^q(\Omega)$ ,  $1 < q < p = \frac{n}{n-\lambda}$ . We also show that if  $h \in L^{1,\lambda}(\Omega)$ ,  $\lambda > n-1$ , then the reduced boundary in  $\Omega$  of a generalized Cheeger set is  $C^{1,\alpha}$  and the singular set has Hausdorff dimension at most  $n-8$  (empty if  $n \leq 7$ ). For the critical case  $h \in L^{1,n-1}(\Omega)$ , we demonstrate that this strong regularity fails. We also prove a structure result for generalized Cheeger sets in  $\mathbb{R}^n$ ; namely, a bounded generalized Cheeger set  $E \subset \mathbb{R}^n$  with  $h \in L^1(\mathbb{R}^n)$  is always pseudoconvex, and any pseudoconvex set is a generalized Cheeger set for some  $h \in L^1(\mathbb{R}^n)$ ,  $h$  nonnegative and not equivalent to zero. A similar structure theorem holds for generalized Cheeger sets in  $\Omega$ .

## 1. INTRODUCTION

In this paper we study the existence and regularity of solutions of the problem

$$(1.1) \quad v_1^{*h} := \sup_{E \subset \bar{\Omega}} V_1^h(E), \quad V_1^h(E) = \frac{\int_E h(x) dx}{\mathcal{H}^{n-1}(\partial^* E)},$$

where  $\Omega$  is an open bounded set with Lipschitz boundary and  $h$  is an integrable function that belongs to the Morrey spaces. We consider in particular the case  $h \geq 0$ . If  $h \equiv 1$ , (1.1) reduces to

$$\sup_{E \subset \bar{\Omega}} \frac{\mathcal{L}^n(E)}{\mathcal{H}^{n-1}(\partial^* E)} := M_1(\Omega),$$

whose solutions are called Cheeger sets. Cheeger established the inequality  $\lambda_1(\Omega) \geq \left(\frac{1}{2M_1(\Omega)}\right)^2$ , where  $\lambda_1(\Omega)$  is the first eigenvalue of the Laplacian under Dirichlet boundary conditions. References to Cheeger sets include Caselles-Chambolle-Novaga [18, 19], Alter-Caselles [2], Figalli-Maggi-Pratelli [25], Parini [42], Alter-Caselles-Chambolle [3], Carlier-Comte-Peyre [16], Carlier-Comte [17], Buttazzo-Carlier-Comte [14], Kawohl-Friedman [33], Kawohl-Novaga [35] and Kawohl-Lachand [34].

A set  $E$  where the supremum (1.1) is attained (i.e.  $v_1^{*h} = V_1^h(E)$ ) is called a *generalized Cheeger set in  $\Omega$* . Indeed, generalized Cheeger sets refer to more general problems of the type  $\sup_{E \subset \bar{\Omega}} \frac{\int_E h_1 dx}{\int_{\partial^* E} h_2 d\mathcal{H}^{n-1}}$ . Generalized Cheeger have been studied when  $h_1 \in L^\infty$  and  $h_2$  continuous (see Buttazzo-Carlier-Comte [14] and the references therein). We will show in this paper that it is natural to study (1.1) with  $h$  belonging to the Morrey spaces (see also Ionescu-Lachand-Robert [32]).

Since a necessary condition for generalized Cheeger sets to exist is that

$$(1.2) \quad \sup_{E \subset \bar{\Omega}} V_1^h(E) < \infty,$$

---

*Date:* June 2015.

*Key words and phrases.* function spaces, geometric measure theory, calculus of variation, minimal surfaces.

we start our study by analyzing the space  $S(\Omega)$ , consisting of all functions  $g$  that satisfy, for each set  $E \subset \mathbb{R}^n$  of finite perimeter,

$$(1.3) \quad \sup \frac{\int_{E \cap \Omega} |g|}{\mathcal{H}^{n-1}(\partial^* E)} < \infty.$$

Indeed,  $h$  satisfies (1.2) if and only if  $h \in S(\Omega)$  (see Lemma 3.6). We note our notation

$$M^p(\Omega) := L^{1,\lambda}(\Omega), \quad \lambda > 0, \quad \lambda = n \left(1 - \frac{1}{p}\right),$$

where  $L^{1,\lambda}(\Omega)$  is the Morrey space introduced in section 2. In this paper we show that  $S(\Omega)$  coincides with the Morrey space  $M^n(\Omega)$  (see Theorem 3.7).

In order to study characterizations of Morrey spaces in terms of  $S(\Omega)$ , we first note that the isoperimetric inequality implies that  $\frac{\int_{E \cap \Omega} |g|}{\mathcal{H}^{n-1}(\partial^* E)}$  is bounded above by  $\frac{\int_{E \cap \Omega} |g| dx}{|E|^{1-\frac{1}{n}}}$ . This observation suggests that  $S(\Omega)$  can be compared to the Morrey space  $L^{1,n-1}(\Omega)$ , whose definition uses cubes  $Q$ , and  $|Q|^{1-\frac{1}{n}}$  instead of  $|E|^{1-\frac{1}{n}}$ . This observation can be made rigorous using the boxing inequality (see Theorem 3.1), which provides a covering theorem of sets of finite perimeter with balls (or cubes), and that allows to control the perimeters of the elements of the cover in a universal way and in terms of the perimeter of the covered set.

We show in this paper the existence of generalized Cheeger sets whenever  $h$  belongs to the Morrey spaces  $M^p(\Omega)$ ,  $p > n$  (see Theorem 4.4). We also study the critical case  $p = n$  (see section 5 and Example 4.5)

Another space that naturally arises in connection to problem (1.1) is  $M_p(\Omega)$  (see definition 2.11), which is the same as the weak  $L^p$  space, denoted as  $L^{p,w}(\Omega)$  (see Remark 2.12). The existence of generalized Cheeger sets for  $h \in L^n(\Omega)$  was proven in Bright-Li-Torres [12]. In particular, note that the spaces  $L^{p,w}(\Omega)$ ,  $p > n$ , are contained in  $L^n(\Omega)$ . The new existence result in this paper generalizes [12] since the spaces  $M^p(\Omega)$ ,  $p > n$ , are not contained in  $L^n(\Omega)$ . This follows from Section 4, where we show that  $M^p(\Omega)$ ,  $p > n$ , can not be embedded into any space  $L^q(\Omega)$ , with  $1 < q < p$ . This also shows that  $M_p(\Omega) \subsetneq M^p(\Omega)$ .

In the second part of this paper we investigate the regularity of generalized Cheeger sets in  $\Omega$ . For convenience, we consider the equivalent problem

$$(1.4) \quad C_0^h := \inf J^h(F), \quad J^h(F) = \frac{P(F)}{\int_F h dx}, \quad F \subset \bar{\Omega} \text{ is a set of finite perimeter,}$$

with  $h \in M^n(\Omega)$ ,  $h \geq 0$  and  $h$  not equivalent to the zero function. We study the regularity of a generalized Cheeger set  $E$  in  $\Omega$  by noticing that  $E$  is also a minimizer of the functional

$$(1.5) \quad I_H(F) = P(F) + \int_F H(x) dx, \quad F \subset \bar{\Omega},$$

with  $H(x) = -C_0^h h(x)$ . The minimization (1.5), with  $\Omega$  replaced by  $\mathbb{R}^n$ , is called the variational mean curvature problem, and  $H$  is called the variational mean curvature of  $E$ . The regularity of this problem, for  $H \in L^p(\Omega)$ , has been studied by several authors including De Giorgi [22], Massari [38, 39], Barozzi-Gonzalez-Tamanini [9], Gonzalez-Massari-Tamanini [28] and Gonzalez-Massari [27]. The regularity of more general classes of quasi minimizers of perimeter was studied in Tamanini [44], Bombieri [15], Paolini [41] and Ambrosio-Paolini [6]. Using the regularity theory for quasi minimizers developed in Tamanini [44], we show that, if  $E$  is a generalized Cheeger set for some  $h \in M^p(\Omega)$ ,  $p > n$ , then  $\partial^* E \cap \Omega$  is a  $C^{1,\alpha}$ -hypersurface and the singular set has dimension at most  $n-8$  (empty if  $n \leq 7$ ). This strong regularity corresponds to the classical results for the variational mean curvature problem mentioned above for  $H \in L^p(\Omega)$ ,  $p > n$ .

We also study the structure of generalized Cheeger sets in  $\mathbb{R}^n$  (i.e.  $\Omega$  replaced by  $\mathbb{R}^n$  in the minimization (1.4)). For the variational mean curvature problem, it is well known that, for any

set of finite perimeter  $E$ , there exists an  $L^1(\mathbb{R}^n)$  function  $H$  such that  $E$  has mean curvature  $H$  (see [9] and [27]). This implies that we can not expect to have regularity for sets with  $L^1$  mean curvature, since a set of finite perimeter can have a wild boundary (see, for example, [37, Example 12.25]). Motivated by the close relationship between our generalized Cheeger set problem and the variational mean curvature problem, we ask if this is also true in our situation, namely, if for any set of finite perimeter  $E$  with  $0 < |E| < \infty$ , there exists an  $L^1$  nonnegative function  $h$  such that  $E$  is a generalized Cheeger set in  $\mathbb{R}^n$  corresponding to this  $h$ . We answer negatively this question by showing a structure theorem (see Theorem 7.7) which gives a necessary and sufficient condition for  $E$  to be a generalized Cheeger set. More specifically, we show that if  $E$  is a generalized Cheeger set in  $\mathbb{R}^n$  for some  $h \in L^1(\mathbb{R}^n)$ , then  $E$  is a pseudoconvex set (see Definition 7.2), and if  $E$  is a pseudoconvex set, then a non negative function in  $L^1(\mathbb{R}^n)$  can be constructed so that  $E$  minimizes  $J^h$  in  $\mathbb{R}^n$ . Theorem 7.7 has some interesting consequences, in particular the fact that each indecomposable component (see Definition 2.9) of a generalized Cheeger set for some  $h \in L^1(\mathbb{R}^n)$  is a generalized Cheeger set for the same  $h$ , and hence each indecomposable component of a pseudoconvex set is also pseudoconvex (see Theorem 8.1). A similar structure theorem holds for generalized Cheeger sets in  $\Omega$  (see Theorem 7.9).

For the variational mean curvature problem, De Giorgi conjectured in 1992 that if  $H \in L^n$ , then the boundary of a minimizer  $E$  of  $I_H$  is locally parametrizable (out of a singular set, if  $n \geq 8$ ) by a bi-lipschitz map defined on an open ball of  $\mathbb{R}^{n-1}$ . De Giorgi also proposed an example of a quasi minimizer in the plane having a singular point at the origin and parametrizable by a bi-lipschitz map. Gonzalez-Massari-Tamanini [28] proved that the example of De Giorgi, whose boundary is the union of two bilogarithmic spirals, is indeed a set with mean curvature in  $L^2(\mathbb{R}^2)$ . The full conjecture is still an open problem, but the case  $n = 2$  has been solved by Ambrosio-Paolini [6, Theorem 5.2]. For  $n > 2$ , Paolini showed in [41] that the boundary of  $E$  is locally parametrizable for any  $\alpha < 1$  with a map  $\tau$  such that both  $\tau$  and  $\tau^{-1}$  are  $C^\alpha$ . This result was extended to quasi minimizers of perimeter by Ambrosio-Paolini [6, Theorem 4.10].

Motivated by the De Giorgi conjecture, we would like to know if the regularity obtained in [41, 6] also holds for generalized Cheeger sets corresponding to our critical case  $h \in M^n(\Omega)$ . Although this assumption on  $h$  is weaker than  $h \in L^n(\Omega)$ , our structure theorem for generalized Cheeger sets (see Theorem 7.7) suggests that one could expect some type of partial regularity in this critical case. In this paper we prove that the strong regularity fails by showing that there exists a function  $h \in M^n(\Omega)$ , and a generalized Cheeger set in  $\Omega$  whose  $\partial E$  contains a large set (of Hausdorff dimension  $n$ ) consisting of points of zero density.

As an application of our results, we consider in the last section the following averaged shape optimization problem (see [12]):

$$(1.6) \quad v^* := \inf_{E \subset \bar{\Omega}} V(E), \quad V(E) = V_2(E) + V_1^g(E) = \inf_{E \subset \bar{\Omega}} \frac{\int_{\partial^* E} f(x, \nu_E(x)) d\mathcal{H}^{n-1}(x) + \int_E g dx}{\mathcal{H}^{n-1}(\partial^* E)},$$

This problem was studied in [12] under the condition  $g \in L^n(\Omega)$ . It was proven there that, if there exists a minimizing sequence  $E_i$  such that  $|E_i| \rightarrow 0$  or  $\mathcal{H}^{n-1}(E_i) \rightarrow \infty$ , then there exists another minimizing sequence consisting of convex polytopes with  $n + 1$  faces shrinking to a point. In this paper we extend this result to the case  $g \in M^p(\Omega)$ ,  $p > n$ , and we show (see Corollary 9.4) that, if there exists a minimizing sequence  $E_i$  satisfying  $\mathcal{H}^{n-1}(\partial^* E_i) \rightarrow \infty$  or  $P(E_i) \rightarrow 0$ , then there exists another minimizing sequence of convex polytopes with  $n + 1$  faces shrinking to a point. We note that this result does not require  $g$  to be negative, although we are interested in the case  $g \leq 0$ , to be consistent with our existence results of generalized Cheeger sets (see Remark 4.7).

We organize the paper as follows. In section 2 we introduce preliminaries needed for the paper. In section 3 we study a characterization of Morrey spaces. In section 4 we prove the existence of generalized Cheeger sets. In section 5 we study the embedding of Morrey spaces into  $L^p$  spaces. In section 6 we study the regularity of generalized Cheeger sets in  $\Omega$ . In section 7 we prove a structure

theorem for generalized Cheeger sets and in section 8 we study the critical case  $h \in M^n(\Omega)$ . Finally, in section 9, we present an application to averaged shape optimization.

## 2. PRELIMINARIES

In this paper we work with sets of finite perimeter. We refer the reader to the standard references by Maggi [37], Ambrosio-Fusco-Pallara [5] and Evans [23] for the relevant definitions and properties of sets of finite perimeter used in this paper. Also,  $\mathcal{H}^{n-1}$  denotes the  $(n-1)$ -dimensional Hausdorff measure in  $\mathbb{R}^n$ , and  $\mathcal{L}^n$  is the Lebesgue measure in  $\mathbb{R}^n$ . We will use the notation  $\mathcal{L}^n(E) = |E|$ . For a set of finite perimeter  $E$ , we use indistinctly the notation  $P(E) = \mathcal{H}^{n-1}(\partial^*E)$ . The convex hull of a set  $E$  is denoted as  $\hat{E}$ .

*Remark 2.1.* Throughout this paper,  $\Omega$  denotes a bounded open set with Lipschitz boundary.

*Remark 2.2.* In particular, we note that we can write  $E \subset \Omega$  or  $E \subset \bar{\Omega}$  since  $\partial\Omega$  is Lipschitz and hence  $E$  is  $\mathcal{L}^n$ -equivalent to  $E \cup \partial\Omega$ .

**Definition 2.3.** A set of finite perimeter  $E$  is said to be *decomposable* if there exists a partition of  $E$  into two measurable sets  $A, B$  with strictly positive measure such that

$$(2.1) \quad P(E) = P(A) + P(B)$$

If no such a partition exists, we call  $E$  a *indecomposable set*.

*Remark 2.4.* The notion of indecomposable set extends the topological notion of connectedness. It can be easily shown that a connected open set of finite perimeter is indecomposable.

**Definition 2.5.** For  $H \in L^1(\mathbb{R}^n)$ , let  $I_H$  be the functional defined as

$$(2.2) \quad I_H(F) = P(F) + \int_F H(x)dx, \quad F \subset \mathbb{R}^n \text{ is a set of finite perimeter.}$$

A set  $E$  of finite perimeter is said to have variational mean curvature  $H$  if  $E$  is a minimizer of (2.2).

*Remark 2.6.* It can be easily seen (by computing the first variation of the functional  $I_H$ ) that if  $E$  has variational mean curvature  $H$ ,  $H$  is continuous at  $x \in \partial E$  and  $\partial E$  is smooth near  $x$ , then the value of the (classical) mean curvature of  $\partial E$  at  $x$  is given by  $-\frac{H(x)}{n-1}$ .

The fact that every set of finite perimeter has a variational mean curvature was observed for the first time in Barozzi-Gonzalez-Tamanini [9]. Moreover, we have the following

**Theorem 2.7.** *Let  $E$  be a set of finite perimeter with  $|E| < \infty$ , then there exists a function  $H_E \in L^1(\mathbb{R}^n)$ , such that  $H_E$  is the variational mean curvature for  $E$ . Furthermore,*

$$(2.3) \quad H_E < 0 \text{ a.e. on } E, H_E > 0 \text{ a.e. on } \mathbb{R}^n \setminus E,$$

and

$$(2.4) \quad \|H\|_{L^1(E)} = P(E), \quad \|H_E\|_{L^1(\mathbb{R}^n)} = 2P(E).$$

*Proof.* See [27, Theorem 2.3] for the proof of (2.4). The property (2.3) is proven in [27, (2.16) and (2.17)].  $\square$

The following theorem by Ambrosio-Caselles-Masnou-Morel [4, Proposition 3.5, Theorem 1 (section 4)], is an important tool we will use in this article.

**Theorem 2.8.** *Let  $E$  be a set of finite perimeter in  $\mathbb{R}^n$ . Then there exists a unique countable family of pairwise disjoint indecomposable sets  $E_i$  such that  $|E_i| > 0$ ,  $E = \cup_i E_i$ ,  $P(E) = \sum_i P(E_i)$ ,  $\partial^*E = \cup_i \partial^*E_i \pmod{\mathcal{H}^{n-1}}$  and  $\mathcal{H}^{n-1}(\partial^*E_i \cap \partial^*E_j) = 0$ . Moreover, if  $F \subset E$  is an indecomposable set then  $F$  is contained  $\pmod{\mathcal{H}^n}$  in some set  $E_i$ .*

*Remark 2.9.* We call each  $E_i$  in the previous theorem a *indecomposable component* of  $E$ .

We now proceed to review the definition of Morrey spaces, which will be shown to be the natural spaces to study the existence and regularity of generalized Cheeger sets in  $\Omega$ . The spaces  $L^{z,\lambda}(\Omega)$  were introduced by Morrey in [40] and are defined as

$$(2.5) \quad L^{z,\lambda}(\Omega) = \left\{ u : \sup_{r>0, x_0 \in \Omega} \frac{1}{r^\lambda} \int_{B_r(x_0) \cap \Omega} |u(y)|^z dy < \infty \right\}$$

When  $\lambda = 0$ , the Morrey space is the same as the usual  $L^z$  space. When  $\lambda = n$ , the spatial dimension, the Morrey space is equivalent to  $L^\infty$ , due to the Lebesgue differentiation theorem. When  $\lambda > n$ , the space contains only the 0 function.

In this paper we study the Morrey space  $L^{1,\lambda}(\Omega)$ , where  $0 < \lambda < n$ . Recall that  $L^{1,\lambda} = \mathcal{L}^{1,\lambda}$  where the latter is the Campanato space. (See Definition 4.1 and [43, Theorem 4.3]). If  $p > 1$  is such that  $\lambda = n(1 - \frac{1}{p})$  we have that  $u \in L^{1,\lambda}(\Omega)$  if and only if

$$\sup_{r>0, x_0 \in \Omega} \frac{1}{|B_r(x_0)|^{1-\frac{1}{p}}} \int_{B_r(x_0) \cap \Omega} |u(y)| dy < \infty,$$

or equivalently,

$$L^{1,\lambda}(\Omega) := \left\{ u \in L^1(\Omega) : \sup \left\{ \frac{\int_{Q \cap \Omega} |u| dx}{|Q|^{1-1/p}} : Q \text{ is a cube} \right\} < +\infty \right\}.$$

**Definition 2.10.**

$$(2.6) \quad M^p(\Omega) := L^{1,\lambda}(\Omega), \quad \lambda = n(1 - \frac{1}{p}), \quad p > 1.$$

We now define the spaces of functions  $M_p(\Omega)$ ,  $p > 1$ , which are the counterparts of the Morrey spaces  $M^p$ ,  $p > 1$ , as follows:

**Definition 2.11.** For  $p > 1$ , let

$$(2.7) \quad M_p(\Omega) := \left\{ g \text{ Lebesgue measurable} : \sup_{A \subset \Omega, A \text{ measurable}} \frac{\int_A |g| dx}{|A|^{1-1/p}} < +\infty \right\}.$$

We note that  $M_p(\Omega) \subset L^1(\Omega)$  by choosing  $A = \Omega$  in the definition above. Moreover, if  $g \in M_p(\Omega)$ , we define

$$(2.8) \quad \|g\|_{M_p(\Omega)} := \sup_{A \subset \Omega, A \text{ measurable}} \frac{\int_A |g| dx}{|A|^{1-1/p}}.$$

*Remark 2.12.* The standard weak  $L^p$  space, denoted as  $L^{p,w}(\Omega)$ , satisfies (see [12, Lemma 8.3]):

$$(2.9) \quad L^{p,w}(\Omega) = M_p(\Omega), \quad p > 1.$$

*Remark 2.13.* Remark 2.12 implies that  $M_p(\Omega)$ ,  $p > 1$ , can be embedded in to any space  $L^q(\Omega)$ , with  $1 \leq q < p$ .

*Remark 2.14.* Notice that

$$(2.10) \quad M_p(\Omega) \subset M_q(\Omega), \quad 1 < q < p,$$

and

$$(2.11) \quad M^p(\Omega) \subset M^q(\Omega), \quad 1 < q < p.$$

We also study in this paper the relationship between  $M_p(\Omega)$  and  $M^p(\Omega)$ . Clearly, we have

$$(2.12) \quad M_p(\Omega) \subset M^p(\Omega), \quad p > 1,$$

and it is natural to ask whether  $M_p(\Omega) = M^p(\Omega)$ , since they are defined in a similar manner. If this is not true, we still ask the question whether  $M^p(\Omega)$ , for  $p$  large enough, can be embedded into  $M_q(\Omega)$ , for some  $q$  close enough to 1. We will see in section 5 that the answer to both questions is false, by showing that  $M^p(\Omega)$ ,  $p > 1$ , cannot be embedded into any space  $L^q(\Omega)$ , for  $1 < q < p$ .

## 3. A CHARACTERIZATION OF MORREY SPACES

Since the necessary condition for generalized Cheeger sets to exist is that

$$(3.1) \quad v_1^{*h} = \sup_{E \subset \Omega} \frac{\int_E h dx}{\mathcal{H}^{n-1}(\partial^* E)} < \infty, \quad h \text{ nonnegative,}$$

in this section we study appropriate spaces for  $h$  to guarantee (3.1).

A key tool that we will use in this paper is the boxing inequality, which was originally due to Gustin [31] (see also [45, Chapter 5, Lemma 5.9.3]).

**Theorem 3.1.** *Let  $n > 1$  and  $0 < \tau < \frac{1}{2}$ . Suppose  $E$  is a set of finite perimeter such that  $\limsup_{r \rightarrow 0} \frac{|E \cap B_r(x)|}{|B_r(x)|} > \tau$  whenever  $x \in E$ . Then there exists a constant  $C(\tau, n)$  and a sequence of closed balls  $\overline{B}_{r_i}(x_i)$  with  $x_i \in E$  such that*

$$(3.2) \quad E \subset \bigcup_{i=1}^{\infty} \overline{B}_{r_i}(x_i)$$

and

$$(3.3) \quad \sum_{i=1}^{\infty} r_i^{n-1} \leq C \mathcal{H}^{n-1}(\partial^* E).$$

*Remark 3.2.* If  $E$  is an open set, the proof of Theorem 3.1 actually shows that we can take  $\tau = \frac{1}{2}$ . Moreover, the covering  $\{B_{r_i}\}$  can be chosen in such a way that

$$(3.4) \quad \frac{|E \cap B_{r_i/5}(x)|}{|B_{r_i/5}(x)|} = \frac{1}{2}$$

We now introduce the following spaces which are related to the functional (3.1).

**Definition 3.3.**

$$(3.5) \quad S(\Omega) := \left\{ g \in L^1(\Omega) : \sup \left\{ \frac{\int_{E \cap \Omega} |g| dx}{P(E)} : E \text{ is a set of finite perimeter} \right\} < +\infty \right\}.$$

and

$$(3.6) \quad S_c(\Omega) := \left\{ g \in L^1(\Omega) : \sup \left\{ \frac{\int_{K \cap \Omega} |g| dx}{P(K)} : K \text{ is a bounded convex set} \right\} < +\infty \right\}.$$

*Remark 3.4.* We define the semi-norm of  $u$  with respect to  $S(\Omega)$  by

$$\|g\|_{S(\Omega)} := \sup \left\{ \frac{\int_{E \cap \Omega} |g| dx}{P(E)} : E \text{ is a set of finite perimeter} \right\}.$$

We similarly define  $\|g\|_{S_c(\Omega)}$ .

*Remark 3.5.* Since a bounded convex set has finite perimeter (see Exercise 15.14 in [37]),  $S(\Omega) \subset S_c(\Omega)$ .

From the definition of  $S(\Omega)$ , we have the following:

**Lemma 3.6.**  *$h \in S(\Omega)$  if and only if (3.1) holds.*

*Proof.* Clearly, if  $h \in S(\Omega)$  then (3.1) holds. We now assume that  $v^{*h} < \infty$ . It suffices to show that  $\frac{\int_{E \cap \Omega} h dx}{P(E)}$  is uniformly bounded for any connected polytope  $E$  with small perimeter. Indeed, we can restrict to small perimeter because  $h \in L^1$ , and we can restrict to connected polytopes because sets of finite perimeter can be approximated by open polytopes (see [37, Remark 13.13]), and each open polytope  $E$  is a countable disjoint union of its connected components  $E_i$ . Moreover, we have

$$(3.7) \quad \frac{\int_{E \cap \Omega} h dx}{P(E)} \leq \sup_i \frac{\int_{E_i \cap \Omega} h dx}{P(E_i)}.$$

Therefore, for any connected polytope  $E$ , by the boxing inequality (see Remark 3.2), there exist cubes  $Q_k$  such that  $E \subset \cup_k Q_k$  with  $\Sigma_k P(Q_k) \leq CP(E)$ . Since  $\Omega$  is bounded and Lipschitz, we can choose  $\{B_{\delta_i}(x_i)\}_{i=1}^N$  a finite covering of  $\partial\Omega$  such that  $\partial\Omega$  is a Lipschitz graph in  $B_{\delta_i}(x_i)$ . Therefore, when  $P(E)$  is small,  $P(Q_k)$  is small and thus  $\text{diam}(Q_k)$  is small. Hence, by the Lebesgue's number lemma, for those  $Q_k$  such that  $Q_k \cap \Omega \neq \emptyset$ ,  $Q_k$  is either contained in  $\Omega$  or in some  $B_{\delta_i}(x_i)$ . Since  $\partial\Omega$  is the graph of a Lipschitz function in  $B_{\delta_i}(x_i)$ , it follows that  $P(Q_k \cap \Omega) \sim P(Q_k)$  if  $Q_k \cap \Omega \neq \emptyset$ . Therefore,

$$(3.8) \quad \frac{\int_{E \cap \Omega} h dx}{P(E)} \leq \sup_k \frac{\int_{Q_k \cap \Omega} h dx}{P(Q_k)}$$

$$(3.9) \quad \sim \sup_k \frac{\int_{Q_k \cap \Omega} h dx}{P(Q_k \cap \Omega)} \leq v^{*h},$$

and thus  $h \in S(\Omega)$ . □

We would like to characterize the space  $S(\Omega)$  in terms of Morrey spaces. It is clear that  $M_n(\Omega) \subset S(\Omega)$ , since  $\|g\|_{S(\Omega)} \leq C(n) \|g\|_{M_n(\Omega)}$ , by the isoperimetric inequality. However, the space  $M_n(\Omega)$  is not the optimal space to guarantee (3.1), see Example 5.1. It turns out that its counterpart  $M^n(\Omega)$  gives the exact characterization of  $S(\Omega)$ . The main result in this section is:

**Theorem 3.7.**

$$(3.10) \quad S(\Omega) = S_c(\Omega) = M^n(\Omega).$$

*Proof.* We note that, for any cube  $Q$ , there exists  $\alpha(n)$  such that  $|Q|^{1-1/n} = \alpha(n)P(Q)$ . Hence  $\frac{\int_{Q \cap \Omega} |g| dx}{|Q|^{1-1/n}} = \frac{\int_{Q \cap \Omega} |g| dx}{\alpha(n)P(Q)}$ , which immediately implies  $S(\Omega) \subset M^n(\Omega)$ .

Suppose now that  $g \in M^n(\Omega)$ . Using the same argument contained in Remark 3.6, in order to obtain  $g \in S(\Omega)$ , it suffices to show that

$$(3.11) \quad \sup \left\{ \frac{\int_{E \cap \Omega} |g|}{P(E)} : E \text{ is an open connected polytope} \right\} < \infty.$$

The boxing inequality (see Remark 3.2) implies that, for any open connected polytope  $E$ , there exists a finite collection of  $n$ -dimensional cubes  $Q_k$  and a universal constant  $C(n)$  which depends only on the dimension  $n$ , such that  $E \subset \cup_k Q_k$  and

$$\Sigma_k P(Q_k) \leq C(n)P(E).$$

Therefore we have the following estimate

$$(3.12) \quad \begin{aligned} \frac{\int_{E \cap \Omega} |g|}{P(E)} &\leq \frac{\Sigma_k \int_{Q_k \cap \Omega} |g|}{\frac{1}{C(n)} \Sigma_k P(Q_k)} \\ &= \frac{C(n) \alpha(n) \Sigma_k \int_{Q_k \cap \Omega} |g|}{\Sigma_k |Q_k|^{\frac{n-1}{n}}} \\ &\leq \alpha(n) C(n) \sup_k \frac{\int_{Q_k \cap \Omega} |g|}{|Q_k|^{\frac{n-1}{n}}} \leq \alpha(n) C(n) \|g\|_{M^n(\Omega)} < \infty, \end{aligned}$$

which shows that  $g \in S(\Omega)$ . Clearly,  $S(\Omega) \subset S_c(\Omega)$ . Suppose now that  $g \in S_c(\Omega)$ . As before, to obtain  $g \in S(\Omega)$ , it suffices to show that

$$(3.13) \quad \sup \left\{ \frac{\int_{E \cap \Omega} |g|}{P(E)} : E \text{ is a connected polytope} \right\} < \infty.$$

This follows by proceeding as in (3.12), since cubes are convex sets.

□

## 4. EXISTENCE OF GENERALIZED CHEEGER SETS

We first introduce the following spaces:

**Definition 4.1.** We define

$$(4.1) \quad \tilde{S}(\Omega) := \left\{ g \in L^1(\Omega) : \limsup_{P(E) \rightarrow 0} \frac{\int_{E \cap \Omega} |g| dx}{P(E)} = 0, E \text{ is a set of finite perimeter} \right\}.$$

and

$$(4.2) \quad \tilde{S}_c(\Omega) := \left\{ g \in L^1(\Omega) : \limsup_{P(K) \rightarrow 0} \frac{\int_{K \cap \Omega} |g| dx}{P(K)} = 0, K \text{ is a bounded convex set} \right\}.$$

*Remark 4.2.* The arguments used in (3.11) imply that

$$\tilde{S}(\Omega) = \left\{ g \in L^1(\Omega) : \limsup_{P(E) \rightarrow 0} \frac{\int_{E \cap \Omega} |g| dx}{P(E)} = 0, E \text{ is an open connected polytope} \right\}.$$

We now proceed to show the relation between the Morrey spaces introduced in section 2 and the spaces  $\tilde{S}(\Omega)$  and  $\tilde{S}_c(\Omega)$ .

**Theorem 4.3.**

$$(4.3) \quad M^p(\Omega) \subset \tilde{S}(\Omega), \quad p > n, \quad \text{and} \quad \tilde{S}(\Omega) = \tilde{S}_c(\Omega).$$

*Proof.* Let  $g \in M^p(\Omega)$ ,  $p > n$ . From Remark 4.2, it suffices to show that

$$(4.4) \quad \frac{\int_{E \cap \Omega} |g| dx}{P(E)} \rightarrow 0,$$

whenever  $P(E) \rightarrow 0$  and  $E$  is an open connected polytope. For such  $E$ , we apply boxing inequality (see Remark 3.2) to obtain cubes  $Q_k$  and a universal constant  $C(n)$  such

$$(4.5) \quad \sum_k P(Q_k) \leq C(n)P(E).$$

Therefore,

$$(4.6) \quad \begin{aligned} |Q_k| &= (\alpha(n)P(Q_k))^{\frac{n}{n-1}} \\ &\leq (C(n)\alpha(n)P(E))^{\frac{n}{n-1}} \end{aligned}$$

Thus, we have the following estimates:

$$(4.7) \quad \begin{aligned} \frac{\int_{E \cap \Omega} |g|}{P(E)} &\leq \frac{\sum_k \int_{Q_k \cap \Omega} |g|}{\frac{1}{C(n)} \sum_k P(Q_k)} \\ &= \frac{C(n)\alpha(n)\sum_k \int_{Q_k \cap \Omega} |g|}{\sum_k |Q_k|^{\frac{n-1}{n}}} \\ &\leq C(n)\alpha(n) \sup_k \frac{\int_{Q_k \cap \Omega} |g|}{|Q_k|^{\frac{n-1}{n}}} \\ &= \alpha(n)C(n) \sup_k \left[ \frac{\int_{Q_k \cap \Omega} |g|}{|Q_k|^{\frac{p-1}{p}}} |Q_k|^{\frac{1}{n} - \frac{1}{p}} \right] \\ &\leq \alpha(n)C(n) \|g\|_{M^p(\Omega)} (C(n)\alpha(n)P(E))^{\frac{n}{n-1}(\frac{1}{n} - \frac{1}{p})}, \quad \text{by (4.6)}. \end{aligned}$$

Since  $\frac{1}{n} - \frac{1}{p} > 0$ ,  $\frac{\int_{E \cap \Omega} |g|}{P(E)} \rightarrow 0$  if  $P(E) \rightarrow 0$ , which implies that  $g \in \tilde{S}(\Omega)$ . Clearly  $\tilde{S}(\Omega) \subset \tilde{S}_c(\Omega)$ . If  $g \in \tilde{S}_c(\Omega)$  then, for any open connected polytope  $E$ , we proceed as before using (4.5) to obtain

$$(4.8) \quad \frac{\int_{E \cap \Omega} |g|}{P(E)} \leq \frac{\sum_k \int_{Q_k \cap \Omega} |g|}{\frac{1}{C(n)} \sum_k P(Q_k)} = C(n) \sup_k \frac{\int_{Q_k \cap \Omega} |g|}{P(Q_k)} \rightarrow 0$$

whenever  $P(E) \rightarrow 0$ , since (4.5) yields  $P(Q_k) \rightarrow 0$ ,  $g \in \tilde{S}_c(\Omega)$ , and the cubes  $Q_k$  are convex sets.  $\square$

In particular, Theorem 4.3 implies the existence of generalized Cheeger sets:

**Theorem 4.4.** (*Existence of generalized Cheeger sets*) *Let  $h \in M^p(\Omega)$ ,  $p > n$ , with  $h \geq 0$  and  $h$  not equivalent to the zero function. A bounded open set  $\Omega$  with Lipschitz boundary contains a generalized Cheeger set maximizing*

$$(4.9) \quad v_1^{*h} = \sup_{E \subset \Omega} \frac{\int_E h(x) dx}{P(E)}$$

*Proof.* We first note that  $v_1^{*h} > 0$ . Let  $E_i$  be a minimizing sequence of (4.9), thus  $P(E_i)$  are uniformly bounded for otherwise we would have  $v_1^{*h} = 0$ . We also have that  $\liminf_{i \rightarrow \infty} P(E_i) := P_\infty > 0$ , for otherwise we would have that, up to a subsequence,  $P(E_i) \rightarrow 0$  and Theorem 4.3 would imply  $v_1^{*h} = 0$ . Therefore, by the standard compactness theorem for sets of finite perimeter there exists a set  $E_0$  such that, up to a subsequence,  $E_i \rightarrow E_0$  in  $L^1(\Omega)$  and  $P(E_i) \rightarrow P_\infty$ . If  $P(E_0) = 0$  then  $|E_0| = 0$  and hence the dominated convergence theorem yields

$$\frac{\int_{E_i} h(x) dx}{P(E_i)} \rightarrow \frac{\int_{E_0} h(x) dx}{P_\infty} = 0,$$

that is,  $v_1^{*h} = 0$ , which is not possible since  $v_1^{*h} > 0$ . We conclude that  $P(E_0) > 0$  and hence

$$(4.10) \quad v_1^{*h} = \lim_{i \rightarrow \infty} \frac{\int_{E_i} h(x) dx}{P(E_i)} = \frac{P(E_0) \int_{E_0} h(x) dx}{P_\infty P(E_0)}.$$

The lower semicontinuity of the perimeter gives  $P(E_0) \leq P_\infty$ , and hence

$$\frac{\int_{E_0} h(x) dx}{P(E_0)} = \frac{P_\infty}{P(E_0)} v_1^{*h} \geq v_1^{*h}.$$

Therefore,  $P(E_0) = P_\infty$  and  $v_1^{*h}$  is attained by  $E_0$  in the maximization (4.9).  $\square$

The following example shows that in the critical case  $h \in M^n(\Omega)$ , there is no definite conclusion as to whether generalized Cheeger set exists.

**Example 4.5.** Let  $\Omega = B_1(0)$ . Let  $h_1(x) = \frac{n-1}{|x|}$  and  $h_2(x) = \frac{n-1}{|x|} - 1$ . Hence,  $h_1, h_2 \geq 0$ , and it is easy to check that  $h_1, h_2 \in M^n(\Omega) \setminus \cup_{q>n} M^q(\Omega)$ . The arguments in [12, Example 7.6] justify the following computation

$$(4.11) \quad \frac{\int_E h_1(x) dx}{P(E)} = \frac{\int_E \operatorname{div} \frac{x}{|x|} dx}{P(E)} = \frac{\int_{\partial^* E} \frac{x}{|x|} \cdot \nu_E(x) d\mathcal{H}^{n-1}(x)}{P(E)} \leq 1,$$

where "=" holds if and only if  $E$  is equivalent to a ball centered at the origin. Hence, any ball  $B_r(0)$  is a maximizer of  $\frac{\int_E h_2(x) dx}{P(E)}$ . Similarly,

$$(4.12) \quad \frac{\int_E h_2(x) dx}{P(E)} = \frac{\int_E \operatorname{div} \frac{x}{|x|} dx}{P(E)} - \frac{|E|}{P(E)} = \frac{\int_{\partial^* E} \frac{x}{|x|} \cdot \nu_E(x) d\mathcal{H}^{n-1}(x)}{P(E)} - \frac{|E|}{P(E)} \leq 1 - \frac{|E|}{P(E)} < 1.$$

We note that  $\lim_{r \rightarrow 0} \frac{\int_{B_r(0)} h_2(x) dx}{P(B_r(0))} = \lim_{r \rightarrow 0} (1 - C(n)r) = 1$ . Therefore, by (4.12), the maximum of the functional  $\frac{\int_E h_2(x) dx}{P(E)}$ , which is 1, cannot be obtained.

We showed the existence of generalized Cheeger sets assuming that  $h \geq 0$ . We now proceed to explain why we can not expect to have existence if this condition is not satisfied.

**Lemma 4.6.** *Let  $g \in L^1(\Omega)$ , and*

$$\Lambda := \{x \in \Omega : \limsup_{r \rightarrow 0} \frac{\int_{B_r(x)} |g|}{r^{n-1}} > 0\},$$

*then  $\dim \Lambda \leq n - 1$ , where  $\dim$  means Hausdorff dimension. If  $g \in M_n(\Omega)$ , then  $\dim \Lambda = 0$ .*

*Proof.* If  $g \in L^1$ , then by [23, Theorem 2.4.3],  $\mathcal{H}^{n-1}(\Lambda) = 0$ , and hence  $\dim \Lambda \leq n - 1$ . For  $g \in M_n(\Omega)$ , since  $M_n(\Omega)$  can be embedded into any  $L^p(\Omega)$ ,  $1 \leq p < n$ , by applying Holder's inequality, it follows that  $\mathcal{H}^{n-p}(\Lambda) = 0$ , for all  $1 \leq p < n$ . Hence,  $\dim \Lambda = 0$ .  $\square$

*Remark 4.7.* Lemma 4.6 implies that if  $h \in L^1(\Omega)$ , and  $h \leq 0$  in a neighborhood of  $x$ , then for  $\mathcal{H}^{n-1}$ -a.e.  $y$  in the neighborhood,  $\liminf_{r \rightarrow 0} \frac{\int_{B_r(y)} h}{\int_{B_r(y)} h} = -\infty$ , and hence a generalized Cheeger set with  $h \leq 0$  in most cases cannot exist.

## 5. ON THE EMBEDDING PROBLEM OF MORREY SPACES INTO $L^p$ SPACES

The Morrey space  $M^p(\Omega)$  is also defined in [26, Page 164]. It has applications in the study of Sobolev spaces and PDE theory. We recall that  $M_p(\Omega)$ ,  $p > 1$  (i.e., the weak  $L^p$  space), can be embedded into any space  $L^q(\Omega)$ ,  $1 \leq q < p$  (see Remark 2.13). Thus, it is natural to ask whether  $M^p(\Omega)$  has the same property, since both are defined in a similar manner. If this is not true, since both  $L^p(\Omega)$  and  $M^p(\Omega)$  become larger as  $p$  gets smaller, we ask the question whether it is possible to embed  $M^p(\Omega)$  into some  $L^q(\Omega)$  for  $p$  is sufficiently large and  $q$  sufficiently close to 1. Clearly,  $M_p(\Omega) \subset M^p(\Omega)$  for any  $p > 1$ , and thus it is also natural to ask whether  $M^p(\Omega) = M_p(\Omega)$ .

In this section, we construct an example which will rigorously show that for any  $\Omega \subset \mathbb{R}^n$  and  $1 < q < p$ , there exists  $g \in M^p(\Omega) \setminus M_q(\Omega)$ . This answers negatively the questions raised in the previous paragraph.

**Example 5.1.** We let  $\Omega$  be an open bounded set with Lipschitz boundary. For any  $1 < q < p$ , there exists  $0 < t < s < 1$  such that  $q = \frac{1}{1-t}$  and  $p = \frac{1}{1-s}$ . Now for such  $t$  and  $s$ , we choose  $\alpha$  such that

$$(5.1) \quad \frac{1-s}{s} < \alpha < \frac{1-s}{s-t}.$$

By the second inequality of (5.1), we have  $\alpha(1-t) < (\alpha+1)(1-s)$ , and hence we can choose  $\beta$  such that

$$(5.2) \quad \alpha(1-t) < \beta \leq (\alpha+1)(1-s).$$

By the first inequality of (5.1),  $(\alpha+1)(1-s) < \alpha$ , and thus (5.2) yields

$$(5.3) \quad \beta < \alpha.$$

Therefore, we can choose  $\gamma$  such that

$$(5.4) \quad 0 < \gamma < \alpha - \beta.$$

We let  $\tilde{Q}_k$  be the  $n$ -dimensional cube with  $|\tilde{Q}_k| = k^{-\gamma-1}$ , then  $\text{diam}(\tilde{Q}_k) \leq 1$  and  $\sum_{k=1}^{\infty} |\tilde{Q}_k| < \infty$ . By the well known Auerbach-Banach-Mazur-Ulam theorem (see [36]), there is a cube  $\mathcal{Q}$ , such that  $\{\tilde{Q}_k\}$  can be put into the cube  $\mathcal{Q}$  in a manner that no two of the cubes  $\{\tilde{Q}_k\}_{k=1}^{\infty}$  intersect. Without loss of generality, up to rescaling the size of the cubes  $\tilde{Q}_k$  by a constant, we may assume that  $\Omega$  contains a neighborhood of  $\mathcal{Q}$ .

We now let  $Q_k$  be the  $n$ -dimensional cube such that  $|Q_k| = k^{-\alpha-1}$ . As  $\gamma < \alpha$ ,  $|Q_k| < |\tilde{Q}_k|$ , and thus we can put  $Q_k$  into  $\tilde{Q}_k$ . We let the center of each  $Q_k$  coincide with the center of each  $\tilde{Q}_k$ , and the faces of each  $Q_k$  be parallel to the faces of the corresponding  $\tilde{Q}_k$ . We now define

$$(5.5) \quad g := k^\beta \text{ on } Q_k \text{ and } g = 0 \text{ otherwise.}$$

Clearly  $g$  is integrable, since  $\|g\|_{L^1(\Omega)} = \sum_k k^{\beta-\alpha-1} < \infty$ . We will show that for any cube  $Q$ ,  $\frac{\int_{Q \cap Q_k} |g|}{|Q|^s} < \infty$ . Given any  $Q$ , we note that there are three possible situations. First, if  $Q$  does not intersect any of the cubes  $Q_k$ , then

$$\frac{\int_{Q \cap \Omega} |g|}{|Q|^s} = 0.$$

Second, if  $Q \subset \tilde{Q}_k$  for some  $k$ , then

$$\begin{aligned} \frac{\int_{Q \cap Q_k} |g|}{|Q|^s} &\leq \frac{k^\beta |Q \cap Q_k|}{|Q \cap Q_k|^s} \\ &= k^\beta |Q \cap Q_k|^{1-s} \\ &\leq k^\beta |Q_k|^{1-s} \\ &= k^{\beta-(\alpha+1)(1-s)} \\ (5.6) \quad &< \infty, \text{ by the right half of (5.2).} \end{aligned}$$

Finally, if  $Q$  is not contained in any of the cubes  $\tilde{Q}_k$ , we let  $I_Q = \{k : |Q \cap Q_k| > 0\}$ . For  $k \in I_Q$ , by the geometry of  $Q_k$  and  $\tilde{Q}_k$ , there exists a universal constant  $C$  such that

$$(5.7) \quad \frac{|Q \cap \tilde{Q}_k|}{|Q \cap Q_k|} \geq C \frac{k^{-\gamma-1}}{k^{-\alpha-1}} = C k^{\alpha-\gamma}.$$

Therefore

$$\begin{aligned} \frac{\int_{Q \cap \Omega} |g|}{|Q|^s} &\leq \frac{\sum_{k \in I_Q} k^\beta |Q \cap Q_k|}{(\sum_{k \in I_Q} |Q \cap \tilde{Q}_k|)^s} \\ &\leq c \frac{\sum_{k \in I_Q} k^\beta |Q \cap Q_k|}{(\sum_{k \in I_Q} k^{\alpha-\gamma} |Q \cap Q_k|)^s}, \quad c = \frac{1}{C^s}, \text{ by (5.7)} \\ &\leq c \frac{\sum_{k \in I_Q} k^\beta |Q \cap Q_k|}{(\sum_{k \in I_Q} k^\beta |Q \cap Q_k|)^s}, \text{ since } \alpha - \gamma > \beta \\ &\leq c \left( \sum_{k \in I_Q} k^\beta |Q \cap Q_k| \right)^{1-s} \\ (5.8) \quad &\leq c \left( \sum_k k^{\beta-\alpha-1} \right)^{1-s} < \infty \end{aligned}$$

Therefore, for any  $n$ -dimensional cube  $Q$ ,

$$\frac{\int_{Q \cap \Omega} |g| dx}{|Q|^{1-1/p}} = \frac{\int_{Q \cap \Omega} |g|}{|Q|^s} < \tilde{C}(n) < \infty.$$

That is,

$$g \in M^p(\Omega).$$

However, let  $E_K = \cup_{k=K}^\infty Q_k$

$$\begin{aligned} \frac{\int_{E_K} |g|}{|E_K|^{1-\frac{1}{q}}} &= \frac{\int_{E_K} |g|}{|E_K|^t} \\ &= \frac{\sum_{k=K}^\infty k^\beta k^{-\alpha-1}}{(\sum_{k=K}^\infty k^{-\alpha-1})^t} \\ &\sim \frac{K^{\beta-\alpha}}{K^{-\alpha t}} \\ &= K^{\beta-\alpha(1-t)} \\ (5.9) \quad &\rightarrow \infty \text{ as } K \rightarrow \infty, \text{ by the first inequality of (5.2).} \end{aligned}$$

Therefore,  $g \notin M_q(\Omega)$ .

*Remark 5.2.* Example 5.1 will also serve as a model example in the study of the critical case  $g \in M^n(\Omega)$  in section 8.

## 6. ON THE REGULARITY OF GENERALIZED CHEEGER SETS FOR $h \in M^p(\Omega)$ , $p > n$ .

In this section, we study the regularity of generalized Cheeger sets in an open bounded Lipschitz domain  $\Omega$ . A *generalized Cheeger set* in  $\Omega$  is a minimizer of the problem

$$(6.1) \quad C_0^h := \inf_{F \subset \bar{\Omega}} J^h(F), \quad J^h(F) = \frac{P(F)}{\int_F h dx}, \quad h \text{ nonnegative.}$$

We recall that  $C_0^h = 1/v_1^{*h}$ .

*Remark 6.1.* From Remark 3.6 and Theorem 3.7 we see that, for  $n = 2$ ,  $C_0^h > 0$  if and only if  $h \in M^2(\Omega)$ , and for  $n \geq 3$  and  $\Omega$  convex,  $C_0^h > 0$  if and only if  $h \in S(\Omega)$ . Since  $S(\Omega) \subset M^n(\Omega)$  we see that, in these cases, a necessary condition to guarantee the existence of generalized Cheeger sets in  $\Omega$  is that  $h \in M^n(\Omega)$ . This motivates our analysis with Morrey spaces.

We observe that, if the set of finite perimeter  $E$  minimizes  $J^h$ , that is,  $J^h(E) = C_0^h$ , then  $E$  is also the minimizer of the restriction of the functional (2.2) to  $\Omega$ , that is,

$$(6.2) \quad I_H(E) \leq I_H(F) = P(F) + \int_F H(x) dx, \quad F \subset \Omega,$$

where  $H(x) = -C_0^h h(x)$ .

Throughout the rest of the article we continue to assume that  $\text{spt} \mu_E = \partial E$  (see [37, Remark 16.11 and Proposition 12.9]).

We now state the first result concerning the regularity of generalized Cheeger sets.

**Theorem 6.2.** *If  $\Omega$  is a Lipschitz domain in  $\mathbb{R}^n$ ,  $h \in M^p(\Omega)$ ,  $p > n$ , and  $J^h(E) = \inf_{F \subset \Omega} J^h(F)$ , where  $J^h$  is defined as above, then  $\Omega \cap \partial^* E$  is a  $C^{1,\alpha}$ -hypersurface, where  $\alpha = \frac{p-n}{2p}$ . Moreover, the singular set has dimension at most  $n - 8$  (and it is empty if  $n \leq 7$ ).*

*Proof.* Let  $C_0^h = \inf_{F \subset \Omega} J^h(F)$ . Clearly  $C_0^h > 0$ . We consider the functional  $I_H(F) = P(F) + \int_F H(x) dx$  with  $H = -C_0^h h(x)$ . We note that  $I_H$  is nonnegative and  $I_H(E) = 0$ , and hence  $E$  is a minimizer of  $I_H$ . Moreover,  $H \in M^p(\Omega)$ .

Let  $F$  be a set such that  $E \triangle F \subset \subset B_r(x) \cap \Omega$ . By the minimality of  $E$ ,

$$(6.3) \quad P(E; B_r(x)) + \int_{E \cap B_r(x)} H \leq P(F; B_r(x)) + \int_{F \cap B_r(x)} H,$$

and using that  $H \in M^p(\Omega)$  we obtain

$$(6.4) \quad P(E; B_r(x)) \leq P(F; B_r(x)) + \int_{B_r(x)} |H| \leq P(F; B_r(x)) + Cr^{n-\frac{n}{p}} = P(F; B_r(x)) + Cr^{2\alpha} r^{n-1},$$

where  $C = \omega_n \|H\|_{M^p(\Omega)}$  and  $0 < \alpha = \frac{p-n}{2p} < \frac{1}{2}$ . It was shown in Tamanini [44] and Bombieri [15] that if a set  $E$  is a quasi minimizer in the sense that

$$(6.5) \quad P(E; B_r(x)) \leq P(F; B_r(x)) + \eta(r)r^{n-1}$$

for all variations  $F$  of the set  $E$  such that  $F \triangle E \subset \subset B_r(x)$  where  $\eta : (0, r_0) \rightarrow [0, \infty)$  satisfies

$$(6.6) \quad \eta(0+) = 0, \quad \frac{\eta(r)}{r} \text{ is increasing on } (0, r_0)$$

and

$$(6.7) \quad \int_0^{r_0} \frac{\sqrt{\eta(r)}}{r} dr < \infty,$$

then  $\partial E$  can be split into the union of a  $C^1$  relatively open hypersurface and a closed singular set with Hausdorff dimension at most  $n - 8$  (empty if  $n \leq 7$ ). Moreover, given  $x \in \partial^* E$ , there exist  $C, r > 0$  such that

$$(6.8) \quad |\nu_E(x) - \nu_E(y)| \leq C \left( \int_0^{|x-y|} \frac{\sqrt{\eta(r)}}{r} + |x-y|^{1/2} \right), \text{ for all } y \in B(x, r) \cap \partial^* E.$$

In our situation, from (6.4) it follows that our generalized Cheeger set  $E$  is a quasi minimizer in the sense of Tamanini (6.5) with  $\eta(r) \leq cr^{2\alpha}$ . Moreover, we note that  $\eta$  satisfies the required conditions (6.6) and (6.7). Furthermore, from (6.8) we conclude that  $\partial E$  can be split into the union of a  $C^{1,\alpha}$  relatively open hypersurface and a closed singular set with Hausdorff dimension at most  $n - 8$  (empty if  $n \leq 7$ ). □

Corollary 4.4 immediately implies the following:

**Theorem 6.3.** *If  $\Omega$  is a Lipschitz domain in  $\mathbb{R}^2$  and  $h \in M^p(\Omega)$ ,  $p > 2$ , then  $\inf_{F \subset \Omega} \frac{P(F)}{\int_F h(x) dx}$  is attained by a set  $E$  of finite perimeter. Moreover,  $\Omega \cap \partial E$  is a  $C^{1,\alpha}$ -hypersurface, where  $\alpha = \frac{p-n}{2p}$ .*

**Corollary 6.4.** *If  $h \in L^p(\Omega)$ ,  $p > n$ , then  $\inf_{F \subset \Omega} \frac{P(F)}{\int_F h(x) dx}$  is attained by a set  $E$  of finite perimeter. Moreover, there exists a closed set  $\Sigma(E)$  with Hausdorff dimension not greater than  $n - 8$  (empty if  $n \leq 7$ ), such that  $(\partial E \cap \Omega) \setminus \Sigma(E)$  is a  $C^{1,\frac{p-n}{2p}}$   $(n-1)$ -dimensional hypersurface.*

*Proof.* That the infimum is attained by a set  $E \subset \Omega$  follows from [12, Corollary 6.2], where the existence for generalized Cheeger sets was proven for  $h \in L^n(\Omega)$ . The regularity follows from the classical theory of variational mean curvatures (see, for example, Gonzalez-Massari [27] and the references therein) or from Theorem 6.2 since  $L^p(\Omega) \subset M^p(\Omega)$ . □

**Corollary 6.5.** *If  $h \in L^n(\Omega)$ , then generalized Cheeger sets  $E$  exist. Moreover, there exists a closed set  $\Sigma(E)$  with Hausdorff dimension not greater than  $n - 8$ , such that  $(\partial E \cap \Omega) \setminus \Sigma(E)$  is a  $C^{0,\alpha}$   $(n-1)$ -dimensional manifold for all  $\alpha < 1$ . If  $n = 2$ , then  $\partial E \cap \Omega$  is a Lipschitz 1-dimensional manifold.*

*Proof.* The existence follows from [12, Corollary 6.2]. The regularity for the critical case  $h \in L^n(\Omega)$  was proven by Ambrosio-Paolini [6, Theorem 4.10, Theorem 5.2] (see also Paolini [41]). □

We finish this section with a discussion about the critical case  $h \in M^n(\Omega)$ . Indeed, motivated by the conjecture of De Giorgi discussed in the introduction (see Ambrosio-Paolini [6]) concerning the regularity of boundaries with prescribed mean curvature in the critical space  $L^n$ , we would like to know if the regularity in Corollary 6.5 holds for generalized Cheeger sets with  $h$  in the critical space  $M^n(\Omega)$ . We first note that a generalized Cheeger set is also a quasi minimizer in the sense of Ambrosio-Paolini [6] since, from (6.5), it follows that

$$(6.9) \quad P(E; B_r(x)) \leq (1 + \omega(r))P(F; B_r(x)),$$

where  $\omega(r) = \frac{c(n)r^{2\alpha}}{1-c(n)r^{2\alpha}}$ ,  $\alpha = \frac{n-p}{2p}$ . However, for the critical case  $p = n$ ,  $\omega(r)$  becomes a constant and hence the condition

$$(6.10) \quad \lim_{r \rightarrow 0} \omega(r) = 0$$

is not satisfied. Thus, we can not proceed as is [6], where this condition was critical to obtain the regularity described in Corollary 6.5. Indeed, the dimension estimates for the singularities of the minimizers in Corollary 6.5 depend heavily on the fact that, if  $h$  is  $L^n$  integrable, then the blow-up limit  $E_\infty$  is a minimal set with mean curvature  $H = 0$  (see [28, Theorem 1.1]). Then, the dimension estimates for the singularities follow from the non-existence of minimizing cones for  $n \leq 7$  and Federer's dimension reduction argument (see, for example, [37, Chapter 28]). However, in order

to show that the blow-up limit has zero mean curvature,  $\omega(r)$  must be infinitesimal, that is, the condition  $w(0+) = 0$  is needed.

The next Example 6.6 shows that the strong regularity described in Theorem 6.2 is not true in general for the critical case  $h \in M^n(\Omega)$ . We will show in Lemma 7.16 that our structure Theorem 7.7 for generalized Cheeger sets implies upper density estimates on points of the boundary of generalized Cheeger sets. The following Example 6.6 also shows that the lower density bounds can fail for  $h \in M^n(\Omega)$ . We will provide in Section 8 a stronger example (see Example 8.4), which shows that, even for  $h \in M_n(\Omega) \subset M^n(\Omega)$ , the set of points where the lower density estimate fails can be very large.

Although Example 8.4 gives more information, Example 6.6 provides motivations for the proof of Theorem 7.7 and gives understanding about their behavior in the critical case  $h \in M^n(\Omega)$ .

**Example 6.6.** This example is a continuation of Example 5.1. Let  $\Omega$  be a bounded open set with Lipschitz boundary and let  $n = 2$ . Let  $s = 1 - \frac{1}{n}$ . Then, for any  $1 < q < n$ , we can choose  $0 < t < s < 1$  with  $t = 1 - \frac{1}{q}$ . As in Example 5.1, we let  $\frac{1-s}{s} < \alpha < \frac{1-s}{s-t}$  and choose  $\beta = (\alpha + 1)(1 - s) = \frac{\alpha + 1}{n}$ . Thus, we choose  $0 < \gamma < \alpha - \beta$  and define  $Q_k$  and  $\tilde{Q}_k$  as in Example 5.1. We now let

$$(6.11) \quad h := \sum_k k^\beta \chi_{C_k},$$

where  $C_k$  is the classical Cheeger set in  $Q_k$ . Proceeding as in Example 5.1,  $h \notin M^q(\Omega)$ ,  $h \in M^n(\Omega)$  and (5.7) holds.

Let  $C_Q$  be the Cheeger constant of the unit cube. Thus, by scaling of perimeter and volume,

$$(6.12) \quad \frac{P(C_k)}{|C_k|} = \inf_{F \subset Q_k} \frac{P(F)}{|F|} = \frac{C_Q}{l(Q_k)},$$

where the function  $l(Q_k)$  denotes the length of the edge of the cube. Since  $l(Q_k) = k^{-\frac{\alpha+1}{n}} = k^{-\beta}$ , we have

$$\frac{P(C_k)}{\int_{C_k} h} = k^{-\beta} \frac{P(C_k)}{|C_k|} = k^{-\beta} \frac{C_Q}{l(Q_k)} = C_Q.$$

Let  $E = \bigcup_k C_k$ , then

$$J^h(E) = \frac{\sum_k P(C_k)}{\sum_k \int_{C_k} h} = C_Q.$$

For any indecomposable set  $F \subset E$ , by Theorem 2.8,  $F \subset Q_k \pmod{\mathcal{H}^n}$  for some  $k$ . Hence

$$(6.13) \quad \frac{P(F)}{\int_F h} = k^{-\beta} \frac{P(F)}{|F|} \geq k^{-\beta} \frac{P(C_k)}{|C_k|} = C_Q.$$

For general  $F \subset E$ , let  $\{F_k\}$  be the indecomposable components of  $F$ . Hence  $F_k \subset E$ ,  $\frac{P(F_k)}{|F_k|} \geq C_Q$ , and hence

$$J^h(F) = \frac{\sum_k P(F_k)}{\sum_k \int_{F_k} h} \geq C_Q.$$

Therefore  $J^h(E) = \inf_{F \subset E} J^h(F)$ . We now claim that  $E$  satisfies (7.6). Indeed, this is because each  $C_k$  is convex (from which we can apply [37, Exercise 15.14]) and the distance between any two of the sets  $C_k$  is large compared to the diameter of each  $C_k$ , and hence, if  $\partial F$  connects two cubes  $Q_i, Q_j$ , the part of  $\partial F$  whose projections to the direction of the edges of each  $Q_k$  intersect  $Q_k$ , but are not contained in any cube  $\tilde{Q}_k$ , has much larger length outside the union of the cubes  $Q_k$ . The same projection argument as in [12, Example 7.3] can be used to verify this argument carefully. Therefore, since  $h \geq 0$  on  $E$  and  $h = 0$  on  $E^c$ , Lemma 7.12 implies that this  $h$  and  $E$  satisfies (7.13). Hence, Proposition 7.10 imply that  $E$  satisfies (7.5), and thus  $E$  also satisfies (7.7). We have shown that  $E$  is a generalized Cheeger set in  $\Omega$  corresponding to the function  $h$  defined in (6.11).

We now proceed to show that the strong regularity of  $E$  fails by finding a point  $x_0 \in \partial E \cap E^0$ . Let  $x_k$  be the center of  $Q_k$ . Since all the cubes  $\tilde{Q}_k$  are contained in a cube  $\mathcal{Q} \subset \subset \Omega$ , we can find  $x_0$  such that, up to a subsequence,  $x_k \rightarrow x_0$ . Clearly  $x_0 \in \text{spt} \|D\chi_E\| = \partial E$ . Let  $Q_r$  be the cube centered at  $x_0$  with edge length  $r$ . Then, for any  $r$ , by the geometry of  $Q_k$  and  $\tilde{Q}_k$  there exists a universal constant  $C_1$  such that  $\frac{|Q_r \cap \tilde{Q}_k|}{|Q_r \cap Q_k|} \geq C_1 k^{\alpha-\gamma}$  (see (5.7)). Let  $I(r) = \{k : Q_r \cap Q_k \neq \emptyset\}$  and  $m(r) = \inf I(r)$ , then  $m(r) \rightarrow \infty$  as  $r \rightarrow 0$ . We have the following

$$\begin{aligned}
\frac{|E \cap B_r|}{r^n} &\sim \frac{|E \cap Q_r|}{|Q_r|} = \frac{\sum_{k \in I(r)} |C_k \cap Q_r|}{\sum_{k \in I(r)} |\tilde{Q}_k \cap Q_r|} \\
&\leq \sup_{k \in I(r)} \frac{|C_k \cap Q_r|}{|\tilde{Q}_k \cap Q_r|} \\
&< \sup_{k \in I(r)} \frac{|Q_k \cap Q_r|}{|\tilde{Q}_k \cap Q_r|} \\
(6.14) \qquad &\leq \frac{1}{C_1} m(r)^{\gamma-\alpha} \rightarrow 0,
\end{aligned}$$

which implies  $x \in E^0$ . We conclude that  $E$  is a generalized Cheeger set in  $\Omega$  corresponding to the critical case  $h \in M^n(\Omega)$  for which the strong regularity fails.

## 7. STRUCTURE OF GENERALIZED CHEEGER SETS

The previous section was concerned with the differentiability properties of the boundaries of generalized Cheeger sets. In this section, we are concerned about the *shape* of the sets, and in particular their property of being pseudoconvex.

We recall that, for the variational mean curvature problem, given *any* set of finite perimeter  $E$ , there exists a function  $H_E \in L^1$  such that  $E$  has mean curvature  $H_E$ . Therefore, since sets of finite perimeter can have very rough boundaries, the same is true for minimizers of this problem. However, for the generalized Cheeger set problem, we will show in Theorem 7.7 that minimizers have the extra property of being pseudoconvex. Before stating the main result of this section, we make some definitions:

**Definition 7.1.** Let  $\Omega \subset \mathbb{R}^n$  be a bounded open set with Lipschitz boundary or  $\Omega = \mathbb{R}^n$ , and let  $E \subset \Omega$  be a bounded set of finite perimeter. We say that  $\tilde{F}$  is a set with least perimeter in  $\Omega$  containing  $E$  if

$$(7.1) \qquad P(\tilde{F}) = \inf\{P(F) : \Omega \supset F \supset E\},$$

or, equivalently

$$P(\tilde{F}) \leq P(F), \text{ for all } \Omega \supset F \supset E.$$

**Definition 7.2.** A bounded set  $E \subset \mathbb{R}^n$  of finite perimeter is called *pseudoconvex* if  $E$  is a set with least perimeter in  $\mathbb{R}^n$  containing  $E$ , that is

$$(7.2) \qquad P(E) = \inf\{P(F) : \mathbb{R}^n \supset F \supset E\}.$$

A pseudoconvex set  $E$  is also called a *subsolution* to the least area problem. If  $E \subset \mathbb{R}^n$  is a bounded set of finite perimeter, then its *minimal hull*  $E_m$  is defined as

$$E_m = \bigcap_{S \in \mathcal{J}_E} S,$$

where  $\mathcal{J}_E$  denotes the family of pseudoconvex sets in  $\mathbb{R}^n$  that contain  $E$ .

*Remark 7.3.* From [8, Proposition 1.1], if each  $E_i$ ,  $i = 1, 2, \dots$ , is a pseudoconvex set, then  $\cap_i E_i$  is also a pseudoconvex set and, if in addition  $E_i \subset E_{i+1}$ , then  $\cup_i E_i$  is pseudoconvex. Hence, if  $E$  is a convex set, then  $E$  is a pseudoconvex set (since a convex set is the intersection of hyperplanes). We

note that the disjoint union of two pseudoconvex sets is not necessarily pseudoconvex (consider for example two disjoint parallel cubes sufficiently close to each other).

*Remark 7.4.* The previous remark implies that  $E_m$  is a pseudoconvex set (see [10, Proposition 2.2]) containing  $E$  and contained in the convex hull of  $E$ ,  $\hat{E}$ . Moreover,  $E_m$  (see [10, Proposition 2.2]) is a set of least perimeter containing  $E$ , that is, (7.1) holds:

$$(7.3) \quad P(E_m) = \min\{P(F) : E \subset F\},$$

which implies

$$(7.4) \quad P(E_m) \leq P(E) \text{ and } P(E_m) \leq P(\hat{E}).$$

We refer the reader to [24, Lemma 4] for the proof of the following theorem, which is very useful in dealing with the case  $n = 2$ . This theorem is not true in higher dimensions, as shown in Remark 7.6.

**Theorem 7.5.** *Let  $E \subset \mathbb{R}^2$  be an indecomposable bounded set. If  $E$  is not equivalent to a convex open set, then there exists a bounded set  $F$  such that  $E \subset F$  and  $P(F) < P(E)$ .*

**7.1. Remark.** *Theorem 7.5 implies that  $\hat{E} = E_m$  in  $\mathbb{R}^2$ . Moreover,  $E_m = E$  if and only if  $E$  is a convex set.*

*Remark 7.6.* We note that Theorem 7.5 is not true for  $n \geq 3$ . In order to see this, we consider, in dimension 3, an open connected set  $E$  such that the perimeter of its convex hull  $\hat{E}$  is strictly greater than the perimeter of  $E$ . Such a set does not exist in dimension 2, but in dimension 3 one may consider a set with the shape of a banana. We consider now the minimal hull  $E_m$  of  $E$ . If Theorem 7.5 were true in dimension 3 then  $E_m$  would be convex, but if  $E_m$  were convex then  $E_m$  would contain  $\hat{E}$  and therefore [37, Exercise 15.14] and (7.4) would yield  $\hat{E} = E_m$ ; that is,

$$P(\hat{E}) = P(E_m) \leq P(E),$$

which contradicts that  $P(\hat{E}) > P(E)$ . Therefore, in dimension 3, when considering a set  $E$  with  $P(\hat{E}) > P(E)$ , the minimal hull  $E_m$  is not convex.

In this section our main result is the following:

**Theorem 7.7.** *(Structure Theorem)*

*Let  $E$  be a bounded set of finite perimeter with  $|E| > 0$ . Then,  $E$  minimizes*

$$(7.5) \quad J^h(E) = \inf_{F \subset \mathbb{R}^n} J^h(F), \quad J^h(F) = \frac{P(F)}{\int_F h(x) dx}$$

*for some  $h \in L^1(\mathbb{R}^n)$ ,  $h \geq 0$ , if and only if  $E$  is pseudoconvex, that is,*

$$(7.6) \quad P(E) = \inf\{P(F) : F \supset E\}.$$

*Moreover, if  $E$  is a minimizer for some  $J^h$  defined on  $\mathbb{R}^n$  then, each indecomposable component  $E_i$  of  $E$  is also a minimizer for the same  $J^h$  and satisfies  $P(E_i) = \inf\{P(F) : E_i \subset F\}$ . In particular, if  $n = 2$ , then  $E_i$  is  $\mathcal{H}^2$ -equivalent to a convex open set.*

*Remark 7.8.* We recall that a set  $E$  satisfying (7.5) is called a *generalized Cheeger set*.

If instead we consider our problem in a bounded Lipschitz domain  $\Omega$ , we similarly have

**Theorem 7.9.** *Let  $E \subset \Omega$  be a set of finite perimeter with  $|E| > 0$ , where  $\Omega$  is a bounded Lipschitz domain. Then,  $E$  minimizes*

$$(7.7) \quad J^h(E) = \inf_{F \subset \Omega} J^h(F), \quad J^h(F) = \frac{P(F)}{\int_F h(x) dx}$$

*for some  $h \in L^1(\Omega)$ ,  $h \geq 0$ , if and only if  $E$  satisfies*

$$(7.8) \quad P(E) = \inf\{P(F) : \Omega \supset F \supset E\}.$$

Moreover, if  $E$  is a minimizer for some  $J^h$  defined on  $\Omega$  then, each indecomposable component  $E_i$  of  $E$  also minimizes the same  $J^h$  and satisfies  $P(E_i) = \inf\{P(F) : \Omega \supset F \supset E_i\}$ . In particular, if  $\Omega$  is convex and  $n = 2$ , then  $E_i$  is  $\mathcal{H}^2$ -equivalent to a convex open set.

In order to prove Theorem 7.7, we need some preliminary results.

**Proposition 7.10.** *Let  $J^h$  be defined as in (7.5). If  $J^h(E) \leq J^h(F)$  for either  $F \subset E$  or  $F \supset E$ , then  $J^h(E) = \inf_{F \subset \mathbb{R}^n} J^h(F)$ .*

*Proof.* We let  $F$  be any set of finite perimeter. Using that  $E$  is a minimizer of (7.7) and the hypothesis, it follows that

$$(7.9) \quad \frac{P(E)}{\int_E h(x)dx} \leq \min \left\{ \frac{P(E \cap F)}{\int_{E \cap F} h(x)dx}, \frac{P(E \cup F)}{\int_{E \cup F} h(x)dx} \right\} \leq \frac{P(E \cap F) + P(E \cup F)}{\int_{E \cap F} h(x)dx + \int_{E \cup F} h(x)dx} \\ = \frac{P(E \cap F) + P(E \cup F)}{\int_E h(x)dx + \int_F h(x)dx}.$$

By the well known inequality (see [4, Proposition 1, Section 1])

$$(7.10) \quad P(E \cap F) + P(E \cup F) \leq P(E) + P(F)$$

we have

$$\frac{P(E)}{\int_E h(x)dx} \leq \frac{P(E) + P(F)}{\int_E h(x)dx + \int_F h(x)dx}.$$

Therefore,

$$\frac{P(E)}{\int_E h(x)dx} \leq \frac{P(F)}{\int_F h(x)dx}$$

which yields  $J^h(E) = \inf_F J^h(F)$ . □

**Lemma 7.11.** *For  $0 < |E| < \infty$ ,  $E$  a set of finite perimeter, if  $h = -H_E \chi_E$  where  $H_E$  is as in Theorem 2.7, and  $\chi_E$  is the characteristic function of  $E$ , then the following is true:*

$$(7.11) \quad J^h(E) = \inf_{F \subset E} J^h(F)$$

*Proof.* From Theorem 2.7,  $H_E < 0$  a.e. on  $E$ , and thus  $h > 0$  a.e. on  $E$ . Also, from Theorem 2.7,

$$\inf_{F \subset \mathbb{R}^n} I_{H_E}(F) = I_{H_E}(E) = P(E) + \int_E H_E = P(E) - \|H_E\|_{L^1(E)} = 0.$$

Hence, in particular,

$$(7.12) \quad P(F) + \int_F H_E(x)dx \geq 0, \text{ for any } F \subset \Omega.$$

If  $F \subset E$  and  $F$  is not equivalent to the empty set, then  $\int_F h(x)dx = \int_F (-H_E(x))dx > 0$ , and (7.12) yields

$$\frac{P(F)}{\int_F h(x)dx} = \frac{P(F)}{\int_F (-H_E(x))dx} \geq 1 = \frac{P(E)}{\|H_E\|_{L^1(E)}} = \frac{P(E)}{\int_E h(x)dx}$$

This finishes the proof. □

**Lemma 7.12.** *Let  $E \subset \mathbb{R}^n$  is a bounded set of finite perimeter. If there exists  $h \in L^1(\mathbb{R}^n)$ ,  $h \geq 0$  a.e., such that*

$$(7.13) \quad J^h(E) = \inf_{\mathbb{R}^n \supset F \supset E} J^h(F).$$

*then  $E$  satisfies condition (7.6). Conversely, if  $E$  satisfies condition (7.6), then (7.13) is true for any  $h \in L^1(\mathbb{R}^n)$ ,  $h \geq 0$  a.e. on  $E$  and  $h = 0$  on  $E^c$ .*

*Proof.* If (7.13) is true, then for any  $F \supset E$ ,

$$(7.14) \quad \begin{aligned} \frac{P(E)}{\int_E h(x)dx} &\leq \frac{P(F)}{\int_F h(x)dx} \\ &\leq \frac{P(F)}{\int_E h(x)dx}, \text{ since } \int_E h(x)dx \leq \int_F h(x)dx. \end{aligned}$$

Therefore,  $P(E) \leq P(F)$  and (7.6) is verified.

If  $E$  satisfies (7.6), we now show that  $E$  satisfies (7.13) for any nontrivial  $h \geq 0$ ,  $h \in L^1(\mathbb{R}^n)$  and  $h \equiv 0$  on  $E^c$ . Indeed, for any such  $h$  we clearly have  $\int_E h(x)dx = \int_F h(x)dx$  for  $F \supset E$ . Therefore, for any  $F \supset E$ , we have

$$\begin{aligned} \frac{P(F)}{\int_F h(x)dx} &= \frac{P(F)}{\int_E h(x)dx} \\ &\geq \frac{P(E)}{\int_E h(x)dx}, \text{ by (7.6),} \end{aligned}$$

which finishes the proof of (7.13).  $\square$

We are now ready to give the proof of Theorem 7.7.

**Proof of Theorem 7.7. :**

If (7.5) is true, then (7.13) is clearly true. By Lemma 7.12, (7.6) is verified. If (7.6) is true, we choose  $h = -H_E \chi_E$ , with  $H_E$  defined as in Theorem 2.7. By Lemma 7.11, (7.11) is satisfied. Since  $h = 0$  on  $E^c$ , Lemma 7.12 shows that (7.13) is true. Hence, by Proposition 7.10, (7.5) is verified and thus we finished the proof.

It remains to show that if  $E$  satisfies the condition (7.5), then each indecomposable component  $E_i$  of  $E$  also satisfies condition (7.5). Indeed, by the minimality of  $E$  and Theorem 2.8, we have the following

$$(7.15) \quad \frac{P(E)}{\int_E h} \leq \inf_i \frac{P(E_i)}{\int_{E_i} h} \leq \frac{\sum_i P(E_i)}{\sum_i \int_{E_i} h} = \frac{P(E)}{\int_E h}.$$

One easily see that "=" holds if and only if

$$(7.16) \quad \frac{P(E_i)}{\int_{E_i} h} = \frac{P(E)}{\int_E h}, \forall i.$$

Therefore for any  $i$ ,  $E_i$  also minimizes  $J^h$ , that is,  $E_i$  satisfies (7.5). This finishes the proof.  $\square$

We note that the proof of Theorem 7.9 is analogous to the proof of Theorem 7.7, with the only extra task of showing that, if  $\Omega$  is convex and  $E \subset \Omega \subset \mathbb{R}^2$  is indecomposable and satisfies (7.8), then  $E$  is equivalent to a convex open set. Indeed, if  $E$  is not convex, by Theorem 7.5, there exists a set of finite perimeter  $F$  such that  $F \supset E$  and  $P(F) < P(E)$ . Since  $\Omega$  is convex, [37, Exercise 15.14] gives  $P(F \cap \Omega) \leq P(F)$ , and thus  $P(F \cap \Omega) < P(E)$ . But  $\Omega \supset F \cap \Omega \supset E$ , and hence (7.8) is not satisfied, which gives a contradiction.

We have the following:

**Proposition 7.13.** *If each  $E_i$ ,  $i = 1, 2, \dots$ , minimizes  $J^h$  in  $\Omega$ , then  $\cup_i E_i$  and  $\cap_i E_i$  (if not  $\mathcal{H}^n$ -equivalent to the empty set) also minimizes  $J^h$  in  $\Omega$ . As a direct consequence, there exists a unique maximal generalized Cheeger set.*

*Proof.* Let  $E$  and  $F$  be generalized Cheeger set for  $J^h$  in  $\Omega$ , then by (7.10), we have

$$\begin{aligned} \frac{P(E)}{\int_E h(x)dx} = \frac{P(F)}{\int_F h(x)dx} &\leq \min \left\{ \frac{P(E \cap F)}{\int_{E \cap F} h(x)dx}, \frac{P(E \cup F)}{\int_{E \cup F} h(x)dx} \right\} \\ &\leq \frac{P(E \cap F) + P(E \cup F)}{\int_{E \cap F} h(x)dx + \int_{E \cup F} h(x)dx} \leq \frac{P(E) + P(F)}{\int_E h(x)dx + \int_F h(x)dx} = \frac{P(E)}{\int_E h(x)dx}. \end{aligned}$$

Therefore,

$$\frac{P(E \cup F)}{\int_{E \cup F} h(x)dx} = \frac{P(E \cap F)}{\int_{E \cap F} h(x)dx} \text{ (if } |E \cap F| > 0) = \frac{P(E)}{\int_E h(x)dx},$$

and hence  $E \cup F$  and  $E \cap F$  (if  $|E \cap F| > 0$ ) are generalized Cheeger set for  $h$ . By induction, generalized Cheeger sets are closed under finite unions and finite intersections (if not equivalent to the empty set). By the lower semicontinuity property of sets of finite perimeter, this is also true for countable unions or countable intersections (if not equivalent to the empty set).  $\square$

Immediately from the proof of Theorem 7.7 and Proposition 7.13, we have the following:

**Corollary 7.14.** *If  $E$  is a generalized Cheeger set in  $\mathbb{R}^n$  ( $\Omega$  resp.) for some  $h$ , then the union of any subcollection of its indecomposable components is still a generalized Cheeger set for the same  $h$ , that is, such union satisfies condition (7.5) ((7.7) resp.).*

*Remark 7.15.* Let  $n = 2$  and  $E$  be an indecomposable minimizer for (7.7). If  $\Omega$  is only a Lipschitz domain, it is not difficult to find an example (for instance, let  $\Omega$  be a glove-like picture with a big hole), such that  $E$  is not convex. Even if we choose a ball  $B \subset \Omega$  nontrivially intersecting  $E$ ,  $E \cap B$  may not be equivalent to a convex open set. However, if  $\Omega$  is a simply connected,  $n = 2$  and  $E$  is an indecomposable subset of  $\Omega$  satisfying (7.7) then, for any convex open subset  $K$  of  $\Omega$  nontrivially intersecting  $E$ , we claim that  $K \cap E$  is equivalent to a convex open set. The proof of this claim follows as in [24, Theorem 1]. The only difference is that we cannot directly apply [24, Lemma 4] because the set with least perimeter containing  $E$  might not be contained in  $\Omega$ . In order to solve this problem we argue as follows. Since  $\Omega$  is simply connected, by [4, Corollary 1] we can assume  $E = \text{int } C^+$  where  $C^+$  is a rectifiable Jordan curve. If our claim were false, then we could find  $a, b \in C^+$  such that the segment  $(a, b)$  is contained in  $\Omega$  but also in the exterior of  $C^+$ . Applying the same argument as in the proof of [24, Lemma 4], we could find  $\Gamma_1, \Gamma_2$  and  $F$  such that  $C^+ = \Gamma_1 \cup \Gamma_2$ ,  $F = \text{int } (\Gamma_1 \cup [a, b])$ ,  $E \subset F$  and  $P(E) > P(F)$ . Since  $\Omega$  is simply connected and  $\Gamma \cup [a, b]$  is a Jordan curve, the Jordan Curve Theorem yields  $F \subset \Omega$ , and thus we obtain a contradiction to (7.8).

The following corollary gives upper density estimates at points in the boundary of generalized Cheeger sets.

**Corollary 7.16.** *If  $E$  minimizes  $J^h$  in  $\Omega$  and  $x \in \partial E$ , then there exists  $c_1(n)$  and  $c_2(n)$  such that*

$$\frac{P(E; B_r(x))}{r^{n-1}} \leq c_1(n), \text{ and } \frac{|E \cap B_r(x)|}{r^n} \leq c_2(n)$$

hold for every  $B_r(x) \subset \subset \Omega$ .

*Proof.* We compare  $E$  with  $E \cup B_r(x)$  and use Theorem 7.9 to deduce  $P(E) \leq P(E \cup B_r(x))$ . Since, for a.e.  $r > 0$ ,  $\mathcal{H}^{n-1}(\partial^* E \cap \partial B_r) = 0$ , from [37, Theorem 16.3] we have

$$(7.17) \quad P(E; B_r(x)) \leq \mathcal{H}^{n-1}(E^0 \cap \partial B_r).$$

This proves the first inequality.

Adding  $\mathcal{H}^{n-1}(E^0 \cap \partial B_r)$  on both sides of (7.17), we have

$$P(E \setminus B_r(x)) \leq 2\mathcal{H}^{n-1}(E^0 \cap \partial B_r).$$

Let  $g(r) = |E \setminus B_r(x)|$ , then  $g'(r) = \mathcal{H}^{n-1}(E^0 \cap \partial B_r)$  for a.e.  $r > 0$ . Hence, the isoperimetric inequality yields  $C(n)g(r)^{\frac{n-1}{n}} \leq 2g'(r)$ , and this implies the upper density estimate for  $\frac{|E \cap B_r(x)|}{r^n}$ .  $\square$

8. THE CRITICAL CASE  $h \in M^n(\Omega)$ 

We now recall the notion of pseudoconvex set (see Definition 7.2). In this section we give an example of a pseudoconvex set  $E$  and  $h \in M_n$  (i.e. the weak  $L^n$  space) such that  $E$  minimizes  $J^h$  but  $|\partial E \cap E^0| > 0$ . That is, the strong regularity described in section 6 fails in this critical case.

The result in Corollary 7.14, which is a direct consequence of Theorem 7.7, is stated in the language of pseudoconvex sets in the following:

**Theorem 8.1.** *Any countable union of indecomposable components of a pseudoconvex set is also pseudoconvex, and in particular each indecomposable component of a pseudoconvex set is still pseudoconvex.*

As a direct consequence, we have the following result:

**Corollary 8.2.** *For any set of finite perimeter  $E$ , there exists a countable family  $F_i$  of disjoint, indecomposable and pseudoconvex sets such that*

$$E \subset \cup_i F_i \pmod{\mathcal{H}^n} \text{ and } \sum_i P(F_i) \leq P(E).$$

*Proof.* Given  $E$ , we consider its minimal hull  $E_m$ . We let  $\{F_i\}$  be the indecomposable components of  $E_m$ . Then, by Theorem 8.1, each  $F_i$  is pseudoconvex and hence Theorem 2.8 and Remark 7.4 yield

$$P(E) \geq P(E_m) = \sum_k P(F_i),$$

from which the corollary is proved.  $\square$

**Lemma 8.3.** *Let  $\{x_j\} \subset \mathbb{R}^n$  be a sequence of points in  $\mathbb{R}^n$  and let  $\{\rho_j\}$  be a decreasing sequence of positive numbers in such a way that*

(i)

$$\sum_j \rho_j \leq 1$$

(ii)

$$B_{\rho_j}(x_j) \cap B_{\rho_i}(x_i) = \emptyset \text{ for all } j \neq i.$$

*Then there exists a sequence  $r_j$  such that for any  $0 < s_j < r_j$ ,*

$$G = \cup_j B_{s_j}(x_j)$$

*is a pseudoconvex set.*

*Proof.* Choose a fixed number  $q \geq 4$  large enough such that

$$nw_n \sum_{j \geq 1} \left( \frac{1}{q^{n-1}} \right)^j < \frac{w_{n-1}}{4^{n-1}}.$$

Let  $r_j = \frac{\rho_j}{q^j}$ . For any  $s_j < r_j$ , let

$$G = \bigcup_{j=1}^{\infty} B_{s_j}(x_j).$$

From Proposition 7.13, in order to show that  $G$  is pseudoconvex it is enough to prove that, for every  $m = 1, 2, \dots$ , the set

$$G_m = \bigcup_{j=1}^m B_{s_j}(x_j)$$

is a pseudoconvex set. It suffices to show if  $F$  is a set with least perimeter containing  $G_m$  (see Definition 7.1), then  $F = G_m$ . To show this, it suffices to show the set  $J := \{j : \partial F \cap \partial B_{\rho_j/2}(x_j) \neq \emptyset\}$  is empty. Indeed, if so, then by the minimum property of  $F$  and the pseudoconvexity of  $B_{s_j}(x_j)$ , it

follows that one of the following situations can occur:

(i):  $F \cap B_{\frac{\rho_j}{2}}(x_j) = B_{s_j}(x_j)$ ,

(ii):  $B_{\frac{\rho_j}{2}}(x_j) \subset F$ .

We let  $J_1 = \{j \in \{1, \dots, m\} : F \cap B_{\frac{\rho_j}{2}}(x_j) = B_{s_j}(x_j)\}$  and define

$$F_1 = F \setminus (\cup_{j \in J_1} B_{s_j}(x_j)).$$

It suffices to show that  $F_1 = \emptyset$ . We note from (ii) that  $\text{dist}(\partial F_1, \cup_{j \notin J_1} B_{s_j}(x_j)) > \delta > 0$ . We let  $\tilde{F} = \cup_{j \in J_1} B_{s_j}(x_j) \cup \lambda F_1$ , where  $\lambda < 1$ . For  $\lambda$  close enough to 1 we have that  $G \subset \tilde{F}$  and

$$\begin{aligned} P(\tilde{F}) &= P(\cup_{j \in J_1} B_{s_j}(x_j)) + P(\lambda F_1) = P(\cup_{j \in J_1} B_{s_j}(x_j)) + \lambda^{n-1} P(F_1) < P(\cup_{j \in J_1} B_{s_j}(x_j)) + P(F_1) \\ &= P(F), \end{aligned}$$

which contradicts the minimality of  $F$ . Thus, we conclude that  $F_1 = \emptyset$ . Hence  $G_m = F$  and thus  $G_m$  is pseudoconvex.

Now we are left to prove  $J = \emptyset$ . If there exists a point  $y_j \in \partial F \cap \partial B_{\frac{\rho_j}{2}}(x_j)$  then, the lower density estimate for perimeter minimizers (see for example [37, Section 17.4]) gives  $P(F; C_j) \geq w_{n-1}(\frac{\rho_j}{4})^{n-1}$ , with  $C_j = B_{\rho_j}(x_j) \setminus B_{s_j}(x_j)$ . Let  $\alpha \geq 1$  be the smallest number in  $J$ . We have, by the choice of  $q$ ,

$$\begin{aligned} P(G_m) - P(F) &\leq \sum_{j \geq \alpha} n w_n s_j^{n-1} - w_{n-1} \left(\frac{\rho_\alpha}{4}\right)^{n-1}, \\ &\leq \sum_{j \geq \alpha} n w_n \left(\frac{\rho_j}{q^j}\right)^{n-1} - w_{n-1} \left(\frac{\rho_\alpha}{4}\right)^{n-1} \\ (8.1) \quad &\leq \rho_\alpha^{n-1} \left( n w_n \sum_{j \geq \alpha} \left(\frac{1}{q^{n-1}}\right)^j - \frac{w_{n-1}}{4^{n-1}} \right) < 0, \end{aligned}$$

and thus we get a contradiction to the minimality of  $F$ . Therefore,  $J = \emptyset$ .  $\square$

We can now give the main example in this section. We are motivated by the question of estimating the size of the points with density zero on the boundary of a generalized Cheeger set. By Theorem 8.1, each indecomposable component of  $E$  is also pseudoconvex. Since a ball is convex, and thus pseudoconvex, hence a starting point is to show the existence of a sequence of disjoint balls  $B_i$  such that  $\partial(\cup_i B_i) \setminus \cup_i (\partial B_i)$  has Hausdorff dimension larger than  $n - 1$ . We will accomplish this by using a classical Cantor-like set,  $C_\alpha$ , which is nowhere dense and has positive measure  $\alpha$ . Then, following [8, Section 3], where a pseudoconvex set with  $|\partial E| > 0$  was constructed, we will show the existence of a pseudoconvex set  $E$  with  $|\partial E \cap E^0| > 0$ .

Although Example 8.4 is written for the case  $n = 2$ , a similar construction in higher dimensions can be done.

**Example 8.4.** This example uses the classical construction of a Cantor-type set  $C_\alpha \subset [0, 1]$  with positive measure  $0 < \alpha < 1$ . At each step  $m = 1, 2, 3, \dots$ , let  $A_{mi}$ ,  $i = 1, 2, \dots, 2^{m-1}$ , be the open intervals removed in the  $m$ -th step, with length  $\frac{1-\alpha}{3^m}$ , and let  $I_{mk}$ ,  $k = 1, 2, \dots, 2^m$ , be the remaining disjoint closed intervals in the  $m$ -th step. Let  $C_\alpha = \cap_{m=1}^\infty \cup_{k=1}^{2^m} I_{mk}$ , where  $C_\alpha$  is the classical Cantor-type set with measure  $\alpha$ . Denote by  $a_{mi}, b_{mk}$  the centers of the intervals  $A_{mi}$ ,  $i = 1, 2, \dots, 2^{m-1}$ , and  $I_{mk}$ ,  $k = 1, 2, \dots, 2^m$ , respectively. Let  $Q_m = \{(a_{mi}, b_{mk}), (b_{mk}, a_{mi}), i = 1, 2, \dots, 2^{m-1}, k = 1, 2, \dots, 2^m\}$  and let  $D = \cup_{m=1}^\infty Q_m = \{x_j : j \in \mathbb{N}\}$ .

Clearly by the construction,  $C_\alpha \times C_\alpha \subset D$ . We can choose  $\rho_j$  small enough such that the balls  $B_{\rho_j}(x_j)$  are pairwise disjoint. We now choose  $s_j$  satisfying Lemma 8.3 and

$$(8.2) \quad s_{j+1} \leq \frac{1}{2} s_j,$$

$$(8.3) \quad \frac{s_j}{\rho_j} \rightarrow 0$$

and

$$(8.4) \quad \sum_j |B_{s_j}(x_j)| < \alpha^2 = |C_\alpha \times C_\alpha| \leq |\bar{D}|.$$

Let

$$E = \cup_j B_{s_j}(x_j)$$

and hence by Lemma 8.3 and (8.4),  $E$  is pseudoconvex and  $|\partial D \setminus E| > 0$ . Clearly,

$$(8.5) \quad \partial D \setminus E \subset \partial E$$

We note that the construction above also holds for any dimension  $n$ , so we use  $n$  instead of 2 in following calculation.

For any ball  $B$  intersecting both  $B_{\rho_j}(x_j)$  and  $B_{s_j}(x_j)$ , there exists a universal constant  $C$  (i.e., it does not depend on  $B$  and  $j$ ), such that

$$(8.6) \quad \frac{|B \cap B_{s_j}(x_j)|}{|B \cap B_{\rho_j}(x_j)|} \leq C \left( \frac{s_j}{\rho_j} \right)^n \rightarrow 0, \text{ as } j \rightarrow \infty,$$

For any  $x \in \partial D \setminus E$ , we will follow as in Example 6.6 to show that  $x \in E^0$ , and hence (8.5) implies  $|\partial E \cap E^0| > 0$ .

Indeed, let  $B_r$  be the ball centered at  $x$ , then as long as  $B_r$  intersect some  $B_{s_k}(x_k)$ ,  $B_r$  intersect  $B_{\rho_k}(x_k)$ . Let  $I(r) = \{k : B_r \cap B_{s_k}(x_k) \neq \emptyset\}$  and  $m(r) = \inf I(r)$ , then  $m(r) \rightarrow \infty$  as  $r \rightarrow 0$ . We have the following

$$\begin{aligned} \frac{|E \cap B_r|}{r^n} &\sim \frac{|E \cap B_r|}{|B_r|} \\ &\leq \frac{\sum_{k \in I(r)} |B_r \cap B_{s_k}(x_k)|}{\sum_{k \in I(r)} |B_r \cap B_{\rho_k}(x_k)|} \\ &\leq \sup_{k \in I(r)} \frac{|B_r \cap B_{s_k}(x_k)|}{|B_r \cap B_{\rho_k}(x_k)|} \\ &\rightarrow 0, \text{ since } m(r) \rightarrow \infty \text{ and (8.6)} \end{aligned}$$

We now let

$$(8.7) \quad h = \sum_k \frac{1}{s_k} \chi_{B_{s_k}(x_k)},$$

and note that this function is supported on  $E$ . Since  $E$  is pseudoconvex then (7.6) holds and thus Theorem 7.7 implies that  $E$  is a generalized Cheeger set in  $\mathbb{R}^n$  for some  $\tilde{h} \in L^1(\mathbb{R}^n)$ ,  $\tilde{h} \geq 0$ . Proceeding as in Example 6.6, it follows that  $E$  is also a generalized Cheeger set corresponding the function  $h$  defined in (8.7). Clearly,  $E$  is also a generalized Cheeger set in any bounded open set with Lipschitz boundary  $\Omega$  that contains  $[0, 1] \times [0, 1]$ . We will show that

$$h \in M_n(\Omega).$$

Since  $M_n(\Omega) = L^{n,w}(\Omega)$ , we need to show that  $t^n |\{ |h| > t \}|$  is bounded uniformly for  $t$  large (since  $\Omega$  has finite measure). Indeed, for any  $t$  large, there exist  $s_k$  such that

$$\frac{1}{s_k} < t \leq \frac{1}{s_{k+1}},$$

and hence by the definition of  $h$  and (8.2),

$$t^n |\{ |h| > t \}| \leq \frac{1}{s_{k+1}^n} \left| \left\{ h > \frac{1}{s_k} \right\} \right| = \frac{1}{s_{k+1}^n} \sum_{i \geq k+1} n w_n s_i^n \leq n w_n \frac{1}{s_{k+1}^n} \frac{s_{k+1}^n}{1 - (\frac{1}{2})^n} \leq 2 n w_n.$$

The previous example shows the following:

**Theorem 8.5.** *For any open bounded set  $\Omega$  with Lipschitz boundary, there exists a generalized Cheeger set  $E \subset \Omega$ , corresponding to  $h \in M_n(\Omega) \subset M^n(\Omega)$ , such that  $|\partial E \cap E^0| > 0$ . Hence, the strong regularity fails in the critical case  $h \in M^n(\Omega)$ .*

## 9. AN APPLICATION TO AVERAGED SHAPE OPTIMIZATION

In Bright-Li-Torres [12], we considered the following averaged shape optimization problem

$$(9.1) \quad v_2^* = \inf_{E \subset \bar{\Omega}} V_2(E), \quad V_2(E) = \frac{\int_{\partial^* E} f(x, \nu_E(x)) d\mathcal{H}^{n-1}(x)}{\mathcal{H}^{n-1}(\partial^* E)},$$

which includes, for example, the minimization of the averaged flux of a physical quantity in the case  $f(x, \nu_E(x)) = \mathbf{F} \cdot \nu_E$ .

The averaged nature of  $V_2(E)$  in (9.1) allows for the optimal value  $v_2^* = \inf_{E \subset \bar{\Omega}} V_2(E)$  to be approximated by a sequence of sets  $E_i$  with  $P(E_i) \rightarrow \infty$  or  $|E_i| \rightarrow 0$ . In the former case we cannot apply the compactness theorem for sets of finite perimeter, and for the latter case the limit of the minimizing sequence degenerates. Also, due to the averaged nature of the problem, we can not guarantee that the functional  $V_2(E)$  is lower semicontinuous. The study of these degenerated cases was initiated in Bright-Torres [13] and continued in Bright-Li-Torres [12]. The nonlinear problem (9.1) was transformed into a linear problem in [12] by introducing the concept of occupational measures (see [12, Definition 2.5]) and the atomic value of  $V_2$  (see [12, see Section 4]). The perturbation of (9.1) with a Cheeger term:

$$(9.2) \quad v^* = \inf_{E \subset \bar{\Omega}} V(E), \quad V(E) = \frac{\int_{\partial^* E} f(x, \nu_E(x)) d\mathcal{H}^{n-1}(x) + \int_E g dx}{\mathcal{H}^{n-1}(\partial^* E)} = V_2(E) + V_1^g(E),$$

was studied in Bright-Li-Torres [12], under the condition that  $g \in L^n(\Omega)$ . In this section we will show that the main results in [12] remains true, for  $n = 2$ , if  $g$  belongs to the Morrey spaces  $M^p(\Omega)$ ,  $p > 2$ .

We first recall (see [12, Definition 2.5]) and [12, see Section 4]) the following definitions:

**Definition 9.1.** We define the atomic value of the minimization problem  $v_2^* = \inf_{E \subset \bar{\Omega}} V_2(E)$  at the point  $x_0 \in \bar{\Omega}$  as

$$(9.3) \quad f_{atom}(x_0) = \inf_{\mu \in P_0(\mathbb{S}^{n-1})} \int_{\mathbb{S}^{n-1}} f(x_0, v) d\mu(v),$$

where

$$(9.4) \quad P_0(\mathbb{S}^{n-1}) = \left\{ \mu \in P(\mathbb{S}^{n-1}) : \int_{\mathbb{S}^{n-1}} v d\mu(v) = \mathbf{0} \in \mathbb{R}^n \right\}.$$

The atomic value of the problem is

$$(9.5) \quad f_{atom} = \min_{x_0 \in \bar{\Omega}} f_{atom}(x_0)$$

We recall the definitions of the spaces  $\tilde{S}(\Omega)$  and  $\tilde{S}_c(\Omega)$  introduced in section 5. We have the following:

**Theorem 9.2.** *Consider the minimization problem  $v^* = \inf_{E \subset \bar{\Omega}} V(E)$  given by*

$$(9.6) \quad V(E) = \frac{\int_{\partial^* E} f(x, \nu_E(x)) d\mathcal{H}^{n-1}(x) + \int_E g dx}{\mathcal{H}^{n-1}(\partial^* E)} = V_2(E) + V_1^g(E)$$

where  $f \in C(\bar{\Omega} \times S^{n-1})$ .

(a) *If  $g \in \tilde{S}_c(\Omega)$  then, for every point  $x_0 \in \bar{\Omega}$ , the atomic value at  $x_0$ ,  $f_{atom}(x_0)$ , can be realized by a sequence of convex polytopes  $\Delta_i \subset \Omega$  with  $n + 1$  faces shrinking to  $x_0$ , in the sense that*

$$\lim_{i \rightarrow \infty} \sup_{y \in \Delta_i} |y - x_0| = 0,$$

and such that

$$\lim_{i \rightarrow \infty} V(\Delta_i) = f_{atom}(x_0).$$

More precisely,  $v^* \leq f_{atom}$ .

(b) If  $g \in \tilde{S}(\Omega)$  then, if there exists a minimizing sequence  $E_i$  such that  $P(E_i) \rightarrow \infty$  or  $P(E_i) \rightarrow 0$ , then  $v^* \geq f_{atom}$ . In this case, from (a),  $v^* = f_{atom}$  and  $v^*$  can be approximated by convex polytopes  $\Delta_i$  with  $n + 1$  faces shrinking to a point  $x_0$ .

*Remark 9.3.* Since  $\Delta_i$  are convex,  $\lim_{i \rightarrow \infty} \sup_{y \in \Delta_i} |y - x_0| = 0$  actually implies that  $P(\Delta_i) \rightarrow 0$ .

**Proof of Theorem 9.2.** Given  $x_0 \in \overline{\Omega}$ , let  $\Delta_i$  be the sequence of convex polytopes constructed in [12, Proposition 4.9]. Notice that [12, Equality (4.9)] yields

$$(9.7) \quad V_2(\Delta_i) \rightarrow f_{atom}(x_0),$$

By remark 9.3, since  $g \in \tilde{S}_c(\Omega)$ , we have that  $\lim_{i \rightarrow \infty} V_1^g(\Delta_i) = 0$ , and hence

$$(9.8) \quad \lim_{i \rightarrow \infty} V(\Delta_i) = f_{atom}(x_0).$$

Since  $f_{atom} = f_{atom}(x_0)$ , for some  $x_0 \in \tilde{\Omega}$ , we have that  $v^* \leq f_{atom}$ , which is (a).

For (b), if  $\{E_i\}$  is a minimizing sequence of (9.9) such that  $P(E_i) \rightarrow \infty$  then  $\frac{\int_{E_i} g dx}{\mathcal{H}^{n-1}(\partial^* E_i)} \rightarrow 0$  and hence  $V_1^g(E_i) \rightarrow 0$ . Also, if  $\lim_{i \rightarrow \infty} P(E_i) = 0$ , the assumption  $g \in \tilde{S}(\Omega)$  implies again that  $V_1^g(E_i) \rightarrow 0$ . Since  $P(E_i) \rightarrow 0$  implies  $|E_i| \rightarrow 0$ , hence the condition of [12, Theorem 2.16] is satisfied. Let  $\mu_1, \mu_2, \dots \in P(\tilde{\Omega} \times \mathbb{S}^{n-1})$  be the corresponding sequence of occupational measures. By compactness there exists a subsequence, denoted again as the full sequence, such that  $\mu_i \xrightarrow{*} \mu_0 \in P(\tilde{\Omega} \times \mathbb{S}^{n-1})$ . Note that  $\mu_0$  is not necessarily an occupational measure corresponding to a set of finite perimeter. Hence, using [12, property (2.4) of occupational measures],

$$\begin{aligned} v^* &= \lim_{i \rightarrow \infty} \frac{1}{P(E_i)} \int_{\partial^* E_i} f(x, \nu_{E_i}(x)) d\mathcal{H}^{n-1}(x) + 0, \text{ since } V_1^g(E_i) \rightarrow 0, \\ &= \lim_{i \rightarrow \infty} \int_{\tilde{\Omega} \times \mathbb{S}^{n-1}} f(x, v) d\mu_i = \int_{\tilde{\Omega} \times \mathbb{S}^{n-1}} f(x, v) d\mu_0 = \int_{\tilde{\Omega}} \left( \int_{\mathbb{S}^{n-1}} f(x, v) d\mu_0^x \right) dp_0, \end{aligned}$$

where  $\mu_0 = p_0 \otimes \mu_0^x$  is the disintegration of the measure  $\mu_0$ . By [12, Theorem 2.16],  $\mu_0^x \in P_0(\mathbb{S}^{n-1})$ , for  $p_0$ -almost every  $x$ . Then, Definition 9.1 implies that the inner integral is bounded from below by  $f_{atom}$ , and, since  $p_0(\tilde{\Omega}) = \mu_0(\tilde{\Omega} \times \mathbb{S}^{n-1}) = 1$ ,  $f_{atom} \leq v^*$ .  $\square$

From Theorem 4.3 and Theorem 9.2 we conclude with the following:

**Corollary 9.4.** Consider the minimization problem  $v^* = \inf_{E \subset \tilde{\Omega}} V(E)$  given by

$$(9.9) \quad V(E) = \frac{\int_{\partial^* E} f(x, \nu_E(x)) d\mathcal{H}^{n-1}(x) + \int_E g dx}{\mathcal{H}^{n-1}(\partial^* E)} = V_2(E) + V_1^g(E)$$

where  $f \in C(\tilde{\Omega} \times \mathbb{S}^{n-1})$ . If  $g \in M^p(\Omega)$ ,  $p > n$ , then, if there exists a minimizing sequence  $E_i$  such that  $P(E_i) \rightarrow \infty$  or  $P(E_i) \rightarrow 0$ , then  $v^* = f_{atom}$  and  $v^*$  can be approximated by convex polytopes  $\Delta_i$  with  $n + 1$  faces shrinking to a point  $x_0$ .

*Remark 9.5.* We note that both Corollary 9.4 and Corollary 4.4 truly generalize the main approximation result in [12, Theorem 5.3] and the existence result in [12, Corollary 6.2] since the spaces  $M^p(\Omega)$ ,  $p > n$ , are not contained in  $L^n(\Omega)$ . Indeed, we have shown in Section 4 that  $M^p(\Omega)$ ,  $p > n$ , can not be embedded into the weak  $L^n(\Omega)$  space,  $M_n(\Omega)$ , which contains  $L^n(\Omega)$ .

#### ACKNOWLEDGEMENT

The authors would like to thank Elisabetta Barozzi, Nicola Fusco and Changyou Wang for useful comments.

## REFERENCES

- [1] A. D. Alexandrov. Convex Polyhedra. *Springer-Verlag, Berlin, 2005. Monographs in Mathematics. Translation of the 1950 Russian edition by N. S. Dairbekov, S. S. Kutateladze, and A. B. Sossinsky.*
- [2] F. Alter and V. Caselles. Uniqueness of the Cheeger set of a convex body. *Nonlinear Analysis*, 70(1):32-44, 2009.
- [3] F. Alter, V. Caselles and A. Chambolle. A characterization of Convex Calibrable Sets in  $\mathbb{R}^n$ . *Mathematische Annalen*, 332:329-366, 2005.
- [4] L. Ambrosio, V. Caselles, S. Masnou and J.M. Morel. Connected components of sets of finite perimeter and applications to image processing. *Journal of the European Mathematical Society*, 3(1): 39–92,2001.
- [5] L. Ambrosio, N. Fusco, and D. Pallara. Functions of Bounded Variation and Free Discontinuity Problems. *Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press: New York*, 2000.
- [6] L. Ambrosio and E. Paolini. Partial regularity for quasi minimizers of perimeter *Ricerche Mat.*, 48:167-186,1999.
- [7] Z. Artstein and I. Bright. Periodic optimization suffices for infinite horizon planar optimal control. *SIAM J. Control Optim.*, 48:4963–4986, 2010.
- [8] E. Barozzi, E. Gonzalez and U. Massari. Pseudoconvex sets. *Ann.Univ.Ferrara.*, 55:23-35, 2009.
- [9] E. Barozzi, E. Gonzalez and I. Tamanini. The mean curvature of a set of finite perimeter. *Proceedings of the American mathematical society*, 99(2):313-316, 1987.
- [10] R.C. Bassanezi and I. Tamanini. Subsolutions to the Least Area Problem and the Minimal Hull of a Bounded Set in  $\mathbb{R}^n$ . *Ann. Univ. Ferrara. Sez. VII-Sc.Mat.*, Vol.XXX: 27-40, 1984.
- [11] P. Billingsley. Convergence of probability measures. *Wiley-Interscience*, 2009.
- [12] I. Bright, Q. Li and Monica Torres. Occupational measures and averaged shape optimization *Submitted*, [www.math.purdue.edu/~torres](http://www.math.purdue.edu/~torres), 2016.
- [13] I. Bright and M. Torres. The integral of the normal and fluxes over sets of finite perimeter. *Interfaces and free boundaries*, 17, 245-259, 2015.
- [14] G. Butazzo, G. Carlier and M. Comte. On the selection of maximal Cheeger sets. *Differential Integral Equations*, 20(9):991-1004, 2007.
- [15] E. Bombieri. Regularity theory for almost minimal currents. *Arch. Rational Mech. Anal.* 78 (1982), 99-130.
- [16] G. Carlier, M. Comte and G. Peyré. Approximation of maximal Cheeger sets by projection. *Math. Model. Number Anal.*, 43(1):139-150, 2008.
- [17] G. Carlier and M. Comte. On a weighted total variation minimization problem. *J. Funct. Anal.*, 250:214-226, 2008.
- [18] V. Caselles, A. Chambolle and M. Novaga. Uniqueness of the Cheeger set of a convex body. *Pacific Journal of Mathematics*, 231(1):77-90, 2007.
- [19] V. Caselles, A. Chambolle and M. Novaga. Some Remarks on uniqueness and regularity of Cheeger sets. *Rend. Sem. Mat. Univ. Padova*, 123, 2010.
- [20] J. Cheeger. A lower bound for the smallest eigenvalue of the Laplacian. *Problems in analysis, Princeton Univ. Press, New Jersey*, 195-199, 1970.
- [21] Su una teoria generale della misura  $(r - 1)$ -dimensionale in uno spazio a  $r$  dimensioni, *Ann.Mat.Pura Appl.* Ser IV 36(1954),191-213
- [22] Frontiere orientate di misura minima, Seminario di Matmatica della Scuola Normale Superiore di Pisa, Editrice Tecnico Scientifica, Pisa, 1961.
- [23] C. Evans and D. Gariepy. Measure Theory and Fine Properties of Functions. *CRC Press, Boca Raton, FL*, 1992.
- [24] A. Ferriero and N. Fusco. A note on the convex hull of sets of finite perimeter in the plane, *discrete and continuous dynamical systems series B*, Volume 11, Number 1, January 2009.
- [25] A. Figalli, F. Maggi and A. Pratelli. A note on Cheeger sets. *Proc. Amer. Math. Soc.*, 137, 2057-2062, 2009.
- [26] David Gilbarg and Neil S. Trudinger. Elliptic partial differential equations of second order, Second edition. *Springer-Verlag, Berlin*, 1983.
- [27] E. Gonzalez and U. Massari. Variational mean curvatures. *Rend. Sem. Mat. Univ. Pol. Torino*, 52(1):1-28, 1994.
- [28] E. Gonzalez, U. Massari and I. Tamanini. Boundaries of prescribed mean curvature. *Rend. Mat. Acc. Lincei*, 4(3):197-206, 1993.
- [29] Loukas Grafakos. Classical Fourier Analysis, Second Edition, Graduate Texts in Mathematics, 249. *Springer, New York*, 2008.
- [30] P. Gruber. Convex and discrete geometry. *Springer*, 2007.
- [31] W. Gustin. Boxing inequalities. *J. Math. Mech.*, 9:229-239, 1960.
- [32] I. R. Ionescu and T. Lachand-Robert. Generalized Cheeger sets related to landslides, *Calculus of Variations and Partial Differential Equations*, 23(2):227–249, 2005.
- [33] B. Kawohl and V. Fridman. Isoperimetric estimates for the first eigenvalue of the  $p$ -Laplace operator and the Cheeger constant. *Comment. Math. Univ. Carolin.*, 44:659-667, 2003
- [34] B. Kawohl and T. Lachand-Robert. Characterization of Cheeger sets for convex subsets of the plane. *Pacific J. Math.*, 225:103-118, 2006

- [35] B. Kawohl and M. Novaga. The  $p$ -Laplace eigenvalue problem as  $p \rightarrow 1$  and Cheeger sets in a Finsler metric, *to appear on J. Convex Analysis*.
- [36] A. Kosinski, A proof of an Auerbach-Banach-Mazur-Ulam theorem on convex bodies. *Colloq.Math.*, **17**(1957), 216-218.
- [37] F. Maggi Sets of finite perimeter and geometric variational problems Cambridge, 2011.
- [38] Umberto Massari. Esistenza e regolarità delle ipersuperfici di curvatura media assegnata in  $\mathbb{R}^n$ . *Arch. Rational Mech. Anal.*,55:357-382, 1974.
- [39] Umberto Massari. Frontiere orientate di curvatura media assegnata in  $L^p$ . *Rend. Sem. Mat. Univ. Padova*, 53:37-52, 1975.
- [40] C.B. Morrey. On the solutions of quasi-linear elliptic partial differential equations. *Trans. Amer. Math. Soc.*, 1:126-166, 1938.
- [41] Paolini. Regularity for minimal boundaries in  $\mathbb{R}^n$  with mean curvature in  $L^n$ , *manuscripta math.* 97,15-35(1998).
- [42] E. Parini. An introduction to the Cheeger problem. *Surveys in Mathematics and its Applications*, 6:9-22, 2011.
- [43] H. Rafeiro, N. Samko and S. Samko. Morrey-Campanato spaces: an overview. (English summary) *Operator theory, pseudo-differential equations, and mathematical physics*, 293–323, Oper. Theory Adv. Appl., 228, Birkhuser/Springer Basel AG, Basel, 2013.
- [44] I. Tamanini. *Regularity Results for Almost Minimal Oriented Hypersurfaces in  $\mathbb{R}^N$* . Quaderni del Dipartimento di Matematica dell'Università di Lecce. Lecce, Università di Lecce, 1984.
- [45] W.P. Ziemer. Weakly Differentiable Functions, Graduate Texts in Mathematics. *Springer-Verlag: New York*, 120, 1989.

[Q. Li] DEPARTMENT OF APPLIED MATHEMATICS, PURDUE UNIVERSITY,  
*E-mail address:* li1380@purdue.edu

[M. TORRES]DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, 150 N. UNIVERSITY STREET, WEST LAYAYETTE,  
 IN 47907-2067, USA. [HTTP://WWW.MATH.PURDUE.EDU/~TORRES](http://www.math.purdue.edu/~torres)  
*E-mail address:* torres@math.purdue.edu