ON THE STRUCTURE OF SOLUTIONS OF MULTIDIMENSIONAL SYSTEMS OF CONSERVATION LAWS

MONICA TORRES

Abstract. We obtain strong traces of solutions of multidimensional systems of conservation laws assuming a weaker regularity property on the entropy solution $\mathbf{u} \in L^\infty(\mathbb{R}^{d+1}, \mathbb{R}^m)$. More precisely, given any entropy function η and any hyperplane $\{(t, x) : x \in \mathbb{R}^d\}$, we show that if $\mathbf{u} \in L^\infty(\mathbb{R}^{d+1}, \mathbb{R}^m)$ is an entropy solution that satisfies the vanishing mean oscillation property on half balls, then $\eta(\mathbf{u})$ has strong traces $\mathcal{H}^d$-almost everywhere on the hyperplane. For the general case, given any set of finite perimeter E and $\boldsymbol{\nu} : \partial^* E \to \mathbb{S}^d$ its inner unit normal and assuming the vanishing mean oscillation property on half balls, we show that the weak trace of the vector field $(\eta(\mathbf{u}), \mathbf{q}(\mathbf{u}))$ defined in Chen-Torres-Ziemer [7] is indeed strong, for any entropy pair $(\eta, \mathbf{q})$.

1. Introduction

In this paper we employ the theory of divergence-measure fields developed in Chen-Torres-Ziemer [7] to obtain strong traces on hyperplanes of entropy solutions of the system

$$\mathbf{u}_t + div_x \mathbf{f}(\mathbf{u}) = 0, \ x \in \mathbb{R}^d \ , \ \mathbf{u} \in L^\infty(\mathbb{R}^{d+1}, \mathbb{R}^m) \ , \tag{1.1}$$

where $\mathbf{f} = (\mathbf{f}^1, \mathbf{f}^2, ..., \mathbf{f}^m)$ and $\mathbf{f}^i : \mathbb{R}^m \to \mathbb{R}^d$. For the general case, given any set of finite perimeter E and $\boldsymbol{\nu} : \partial^* E \to \mathbb{S}^d$ its inner unit normal we show that the weak trace of the vector field $(\eta(\mathbf{u}), \mathbf{q}(\mathbf{u}))$ defined in [7] is indeed strong, for any entropy pair $(\eta, \mathbf{q})$. Our results assume a vanishing mean oscillation property of $\mathbf{u}$ on half balls (see (3.1)), which was proven to be true for $m = 1$ in De-Lellis-Otto-Wesdickenberg [9] (see (3.2)).

More precisely, given any entropy function η, we prove that $\eta(\mathbf{u})$ has strong traces $\mathcal{H}^d$-almost everywhere on any hyperplane $\{(t, x) : x \in \mathbb{R}^d\}$ (see Theorem 3.9). For the general case, given any set of finite perimeter E and $\boldsymbol{\nu} : \partial^* E \to \mathbb{S}^d$ its inner unit normal , we show the existence of the strong trace of $(\eta(\mathbf{u}), q(\mathbf{u})) \cdot \boldsymbol{\nu}(\mathbf{z})$ at $\mathbf{z}$, for every entropy pair $(\eta, \mathbf{q})$ and $\mathcal{H}^d$-almost every point $\mathbf{z} \in \partial^* E$ (see Theorem 4.6).

Vasseur [18] obtained strong traces of entropy solutions of multidimensional scalar conservation laws on any Lipschitz deformable boundary. The regularity of entropy solutions of multidimensional scalar conservation laws was also studied in De-Lellis-Otto-Wesdickenberg [9], were it was shown that u has the structure of a BV function in the sense that the *shock waves* are supported on a codimension-one rectifiable set where u has strong traces. In both [9] and [18], the analysis is done within the framework of the kinetic formulation of conservation laws and assuming the following genuine nonlinearity condition on the flux function $\mathbf{f}$:

$$\mathcal{L}^1(\{v \in \mathbb{R} | \mathbf{f}'(v) \cdot \xi = 0\}) = 0, \text{ for all } \xi \in \mathbb{S}^d. \tag{1.2}$$

For $m = 1$, $d = 1$ and general flux function f, strong traces for a class of functionals of the entropy solution, on any Lipschitz deformable boundary, were obtained in Kwon-Vasseur [19] (see also [13]).

For $d > 1$,$m = 1$ and assuming only continuity of the flux vector $\mathbf{f}$, Panov [14] obtained strong traces of entropy solutions on the hyperplane $\{t = 0\}$ using techniques of H-measures. We also refer to Chen-Rascle [5], where the strong trace at $\{t = 0\}$ was proven for $m = 1$, $d = 1$ using compensated compactness results.

In this paper we obtain the desired strong traces exploiting the connection between the entropy inequality and divergence-measure fields (see §2.1). In particular, we use the Gauss-Green formula for divergence-measure fields developed in Chen-Torres-Ziemer [7] to show that the weak trace of the vector field

$$\boldsymbol{F}_{\mathbf{u}}^{\eta,\mathbf{q}}(t,x) := (\eta(\mathbf{u}(t,x)), \mathbf{q}(\mathbf{u}(t,x))),$$

defined in [7] is indeed an strong trace. Our results improve the regularity of entropy solutions outside the shock waves obtained in De Lellis-Otto-Westdickenberg [9], where it was shown that u has only the vanishing mean oscillation (VMO) property outside the *shock waves*.

2. Gauss-Green formula for bounded divergence-measure fields

In this section we first introduce our definitions and then we present the Gauss-Green formula that will be used to obtain our main results, Theorem 3.9 and Theorem 4.6.

2.1. **Definition.** A Radon measure on Ω is a signed regular Borel measure whose total variation on each compact set $K \Subset \Omega$ is finite, i.e. $\|\mu\|(K) < \infty$. The space of Radon measures supported on an open set Ω is denoted by $\mathcal{M}(\Omega)$.

2.2. **Definition.** We will use the notation

$$N := d + 1,$$

and

$$\mathbf{x} := (t, x)$$

where $x \in \mathbb{R}^d$.

2.3. **Definition.** Let Ω be an open set. A vector field $\boldsymbol{F} \in L^p(\Omega; \mathbb{R}^N)$, $1 \le p \le \infty$, is called a *divergence-measure field*, written as $\boldsymbol{F} \in \mathcal{DM}^p(\Omega)$, if $div\,\boldsymbol{F}$, in the sense of distributions, is a (signed) Radon measure with finite total variation on Ω. Furthermore, $\boldsymbol{F}$ is called a $\mathcal{DM}^p_{loc}(\mathbb{R}^N)$ if $\boldsymbol{F} \in \mathcal{DM}^p(D)$, for any bounded open set $D \subset \mathbb{R}^N$.

2.4. **Definition.** For every $\alpha \in [0, 1]$ and every $\mathcal{L}^N$-measurable set $E \subset \mathbb{R}^N$, define

$$E^\alpha := \{\mathbf{y} \in \mathbb{R}^N \, : \, D(E, \mathbf{y}) = \alpha\}, \tag{2.1}$$

where

$$D(E,\mathbf{y}) := \lim_{r\to 0} \frac{|E\cap B_r(\mathbf{y})|}{|B_r(\mathbf{y})|}. \tag{2.2}$$

Then E^α is the set where E *has density* α. We define the *measure-theoretic boundary* of E, $\partial^m E$, as

$$\partial^m E := \mathbb{R}^N \setminus (E^0 \cup E^1). \tag{2.3}$$

2.5. **Definition.** A function $u\colon \Omega \to \mathbb{R}$ is called a function of bounded variation if each partial derivative of u is a Radon measure with finite total variation in Ω. Notationally, we write $u \in BV(\Omega)$. Let $E \Subset \Omega$ be an $\mathcal{L}^N$-measurable subset. We say that E is a *set of finite perimeter* if $\chi_{_E} \in BV(\Omega)$. Consequently, if E is a set of finite perimeter, then $\nabla\chi_{_E}$ is a (vector-valued) Radon measure whose total variation is denoted by $\|\nabla\chi_{_E}\|$.

2.6. **Definition.** Let $E \Subset \Omega$ be a set of finite perimeter. The *reduced boundary* of E, denoted as $\partial^* E$, is the set of all points $\mathbf{y} \in \Omega$ such that

(i) $\|\nabla\chi_{_E}\|\,(B_r(\mathbf{y})) > 0$ for all $r > 0$;
(ii) The limit $\boldsymbol{\nu}_E(\mathbf{y}) := \lim_{r\to 0} \frac{\nabla\chi_{_E}(B_r(\mathbf{y}))}{\|\nabla\chi_{_E}\|(B_r(\mathbf{y}))}$ exists and $|\boldsymbol{\nu}_E(\mathbf{y})| = 1$.

The unit vector, $\boldsymbol{\nu}_{_E}(\mathbf{y})$, is called the *measure-theoretic interior unit normal to* E *at* $\mathbf{y}$ (we sometimes write $\boldsymbol{\nu}$ instead of $\boldsymbol{\nu}_E$ for notational simplicity). In view of the following, we see that $\boldsymbol{\nu}$ is aptly named because $\boldsymbol{\nu}$ is the interior unit normal to E provided that E (in the limit and in measure) lies in the appropriate half-space determined by the hyperplane orthogonal to $\boldsymbol{\nu}$; that is, $\boldsymbol{\nu}$ is the interior unit normal to E at $\mathbf{y}$ provided that

$$D(\{\mathbf{x} : (\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\nu} > 0, \mathbf{x} \notin E\} \cup \{\mathbf{x} : (\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\nu} < 0, \mathbf{x} \in E\},\, \mathbf{y}) = 0. \tag{2.4}$$

The following results concerning the structure of sets of finite perimeter were proved by De Giorgi (see [2], Theorem 3.59, [20], Theorem 5.7.3 and [12], Chapter 3):

2.7. **Theorem.** *Let E be a set of finite perimeter and $\mathbf{y} \in \partial^* E$. Let $\Pi^+ := \{\mathbf{x} \in \mathbb{R}^N : \boldsymbol{\nu}(\mathbf{y})\cdot(\mathbf{x}-\mathbf{y}) > 0\}$ and $\Pi^- := \{\mathbf{x} \in \mathbb{R}^N : \boldsymbol{\nu}(\mathbf{y})\cdot(\mathbf{x}-\mathbf{y}) < 0\}$. For $r > 0$ define*

$$E_r := \{\mathbf{x} \in \mathbb{R}^N : r(\mathbf{x}-\mathbf{y}) \in E\}.$$

Then

(1) As $r \to 0$ the set E_r converges to Π^+, and moreover, for every set A such that $\mathcal{H}^{N-1}(\partial A \cap \partial\Pi^+) = 0$,

$$\lim_{r\to 0} \|\nabla\chi_{E_r}\|\,(A) = \|\nabla\chi_{\Pi^+}\|\,(A) = \mathcal{H}^{N-1}(A\cap\partial\Pi^+).$$

(2) $\lim_{r\to 0} r^{-N}|E\cap B_r(\mathbf{y})\cap\Pi^-| = 0,$
(3) $\lim_{r\to 0} r^{-N}|(\mathbb{R}^N\setminus E)\cap B_r(\mathbf{y})\cap\Pi^+| = 0.$

(4) The reduced boundary of E, $\partial^* E$, is an $(N-1)$-rectifiable set which means that there exists a countable family of C^1-manifolds M_k of dimension $N-1$ and a set $\mathcal{N}$ of $\mathcal{H}^{N-1}$ measure zero such that

$$\partial^* E \subset \big(\bigcup_{k=1}^{\infty} M_k\big) \bigcup \mathcal{N}.$$

(5) The generalized gradient of $\chi_{_E}$ enjoys the following basic relationship with $\mathcal{H}^{N-1}$:

$$\|\nabla \chi_{_E}\| = \mathcal{H}^{N-1} \llcorner \partial^* E$$

(6)

$$\lim_{r \to 0} \frac{\|\nabla \chi_E\| \, (B_r(\mathbf{y}))}{\alpha(N-1) r^{N-1}} = 1,$$

where $\alpha(N-1)$ is the Lebesgue measure of the unit ball in $\mathbb{R}^{N-1}$.

The following result is due to Federer (see also [20], Lemma 5.9.5. and [2], Theorem 3.61):

2.8. **Theorem.** *If $E \Subset \Omega$ is a set of finite perimeter, then*

$$\partial^* E \subset E^{\frac{1}{2}} \subset \partial^m E, \quad \mathcal{H}^{N-1}(\Omega \setminus (E^0 \cup \partial^* E \cup E^1)) = 0. \tag{2.5}$$

In particular, E has density either 0 or 1/2 or 1 at $\mathcal{H}^{N-1}$-a.e. $\mathbf{x} \in \Omega$ and $\mathcal{H}^{N-1}$-a.e. $\mathbf{x} \in \partial^* E$ belongs to $\partial^m E$.

2.9. *Remark.* In view of Definition 2.5, (5) and (2.5) it is clear that if $E \Subset \Omega$ is a set of finite perimeter then $\mathcal{H}^{N-1}(\partial^m E) < \infty$. Conversely, it was proved by Federer (see [11], 4.5.11) that if $\mathcal{H}^{N-1}(\partial^m E) < \infty$ then E is a set of finite perimeter.

We will refer to the sets E^0 and E^1 as the *measure-theoretic exterior and interior* of E. We note that, in general, the sets E^0 and E^1 do not coincide with the topological exterior and interior of the set E. We also note that (2.5) implies, for any set $E \Subset \Omega$ of finite perimeter,

$$\Omega = E^1 \cup \partial^* E \cup E^0 \cup \mathcal{N}$$

where $\mathcal{H}^{N-1}(\mathcal{N}) = 0$.

From Definition 2.5 it follows that a set of finite perimeter E may be altered by a set of $\mathcal{L}^N$- measure zero and still determine the same essential boundary $\partial^m E$. Throughout, we will choose a prefered representative for E and thereby adopt the following convention

2.10. **Definition.** $E := E^1 \cup \partial^* E.$

We will use the following Gauss-Green formula proved in Chen-Torres [6] and Chen-Torres-Ziemer [7] (see also Silhavy [17]):

2.11. **Theorem.** *Let $\boldsymbol{F} \in \mathcal{DM}^{\infty}_{loc}(\Omega, \mathbb{R}^N)$ and let $E \Subset \Omega$ be a bounded set of finite perimeter. Then, there exist functions $\mathcal{F}_i \cdot \boldsymbol{\nu} \in L^{\infty}(\partial^* E)$ and $\mathcal{F}_e \cdot \boldsymbol{\nu} \in L^{\infty}(\partial^* E)$ such that*

$$\int_{E^1} \operatorname{div}(\varphi \boldsymbol{F}) = -\int_{\partial^* E} \varphi(\mathcal{F}_i \cdot \boldsymbol{\nu})(\mathbf{y}) d\mathcal{H}^{N-1}(\mathbf{y})$$

and

$$\int_E div\,(\varphi \boldsymbol{F}) = -\int_{\partial^* E} \varphi(\mathcal{F}_e \cdot \boldsymbol{\nu})(\mathbf{y}) d\mathcal{H}^{N-1}(\mathbf{y}),$$

for every $\varphi \in C_0^\infty(\Omega)$*. Moreover,* $\|\mathcal{F}_i \cdot \boldsymbol{\nu}\|_\infty \leq \|\boldsymbol{F}\|_\infty$ *and* $\|\mathcal{F}_e \cdot \boldsymbol{\nu}\|_\infty \leq \|\boldsymbol{F}\|_\infty$.

We have the following product rule for bounded divergence-measure fields ([7]):

2.12. **Theorem.** *Let* $\boldsymbol{F} \in \mathcal{DM}^\infty(\Omega)$ *and* $g \in BV(\Omega)$ *bounded with compact support. Then*

$$div\,(g\boldsymbol{F}) = g^*\, div\, \boldsymbol{F} + \overline{\boldsymbol{F} \cdot \nabla g}, \tag{2.6}$$

where g^* *is the precise representative of* g*,* $\overline{\boldsymbol{F} \cdot \nabla g}$ *is the weak* limit of the measures* $\boldsymbol{F} \cdot \nabla g_k$*, and* g_k *is a sequence of mollifications of* g*.*

2.13. *Remark.* If $\boldsymbol{F} \in \mathcal{DM}^\infty(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$ then $div\,(\varphi \boldsymbol{F}) = \varphi div\, \boldsymbol{F} + \boldsymbol{F} \cdot \nabla \varphi$. If $\boldsymbol{F} \in \mathcal{DM}^\infty_{loc}(\Omega)$ then $div\, \boldsymbol{F} << \mathcal{H}^{N-1}$ (see [7], Lemma 2.25).

2.1. Entropy solutions and divergence-measure fields.

2.14. **Definition.** Let $\mathcal{P}$ denote the set of all pairs $(\eta, \mathbf{q})$ such that $\eta : \mathbb{R}^m \to \mathbb{R}$ is convex, $\mathbf{q} \in C^{2,1}(\mathbb{R}^m, \mathbb{R}^d)$ and

$$\nabla \mathbf{q}_k(\mathbf{u}) = \nabla \eta(\mathbf{u}) \nabla \mathbf{f}_k(\mathbf{u}), \quad k = 1, 2, ..., d. \tag{2.7}$$

The pair $(\eta, \mathbf{q})$ is called a convex entropy-entropy pair.

A bounded entropy solution $\mathbf{u} \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R}^m)$ of (1.1) is characterized by the entropy inequality

$$\eta(\mathbf{u})_t + div_x \mathbf{q}(\mathbf{u}) \leq 0 \text{ in } \mathcal{D}'_{t,x} \tag{2.8}$$

for any convex entropy-entropy pair. If we define

$$\boldsymbol{F}^{\eta,\mathbf{q}}_{\mathbf{u}}(t, x) := (\eta(\mathbf{u}(t, x)), \mathbf{q}(\mathbf{u}(t, x))),$$

then the entropy inequality (2.8) and Riesz representation theorem imply (see [10], Corollary 1, Page 53) that there exists a measure $\mu_{\eta,\mathbf{q}} \in \mathcal{M}(\mathbb{R}_+ \times \mathbb{R}^d)$ such that

$$div_{t,x} \boldsymbol{F}^{\eta,\mathbf{q}}_{\mathbf{u}} = \mu_{\eta,\mathbf{q}}.$$

2.15. *Remark.* In this paper we consider the entropy solution $\mathbf{u}$ of (1.1) defined in all $\mathbb{R}^{d+1}$. This can be done in view of the extension theorems for divergence-measure fields proved in Chen-Torres-Ziemer [7] (Section 8). Indeed, setting $\mathbf{u} = 0$ on $\mathbb{R}_- \times \mathbb{R}^d$ we have $\mathbf{u}$ defined in the whole space $\mathbb{R}^{d+1}$ and

$$\boldsymbol{F}^{\eta,\mathbf{q}}_{\mathbf{u}}(t, x) := (\eta(\mathbf{u}(t, x)), \mathbf{q}(\mathbf{u}(t, x))), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d$$

satisfies:

$$\boldsymbol{F}^{\eta,\mathbf{q}}_{\mathbf{u}} \in \mathcal{DM}^\infty_{loc}(\mathbb{R}^N). \tag{2.9}$$

3. Strong traces for multidimensional systems of conservation laws on hyperplanes

In this section we define strong traces for $\eta(\mathbf{u})$ on any hyperplane parallel to $\{t = 0\}$, assuming property (3.1) below. We begin with some definitions:

3.1. **Definition.** We denote the open ball of radius r and center (τ, y) as $B_r(\tau, y)$. For every (τ, y) we denote by $B_r^+(\tau, y)$ as the intersection of $B_r(\tau, y)$ with the set $\Pi^{\tau,y} := \{(t, x) : t > \tau\}$. We also define the cylinder

$$C_r^+(\tau, y) := \mathcal{B}_r(\tau, y) \times (0, r),$$

where $\mathcal{B}_r(\tau, y)$ denotes the intersection of $B_r(\tau, y)$ with the set $\partial\Pi^{\tau,y}$.

3.2. *Remark.* Since $B_r^+(\tau, y)$ can be inscribed in $C_r^+(\tau, y)$, then the results in this section can be stated for cylinders or balls.

3.3. **Definition.** We denote by $\overline{\mathbf{u}}_r(\tau, y)$ the vector in $\mathbb{R}^m$ which is the average of $\mathbf{u}$ over the half ball $B_r^+(\tau, y)$.

3.4. **Definition.** We say that $\mathbf{u}$ satisfies the vanishing mean oscillation property on $\partial\Pi^{\tilde{\tau},\tilde{y}}$ for half balls if, for any continuous $\mathbf{q} \in C(\mathbb{R}^m, \mathbb{R}^d)$:

$$\lim_{r\to 0} \frac{1}{r^{d+1}} \int_{B_r^+(\tau,y)} |\mathbf{q}(\mathbf{u}(t,x)) - \overline{\mathbf{q}(\mathbf{u})}_r(\tau,y)|\, dtdx = 0, \tag{3.1}$$

for $\mathcal{H}^d$-almost every $(\tau, y) \in \partial\Pi^{\tilde{\tau},\tilde{y}}$, where $\overline{q(\mathbf{u})}_r(\tau, y)$ is the vector in $\mathbb{R}^d$ which is the average of $\mathbf{q}(\mathbf{u})$ over the half ball $B_r^+(\tau, y)$.

3.5. *Remark.* For the scalar case (i.e. $m = 1$) and for any hyperplane $\Pi^{\tilde{\tau},\tilde{y}}$, property (3.1) follows from De Lellis-Otto-Westdickenberg [9], as we proceed to show next:

3.6. **Theorem** (De Lellis-Otto-Westdickenberg [9])**.** *Let $(\tilde{\tau}, \tilde{y}) \in \mathbb{R}^{d+1}$. If $\mathbf{f} \in C^{2,1}$ satisfies* (1.2) *and if $u \in L^\infty(\mathbb{R}^{d+1}, \mathbb{R})$ satisfies the entropy inequality* (2.8)*, then for $\mathcal{H}^d$-almost every $(\tau, y) \in \partial\Pi^{\tilde{\tau},\tilde{y}}$,*

$$\lim_{r\to 0} \frac{1}{r^{d+1}} \int_{B_r^+(\tau,y)} |u(t,x) - \overline{u}_r(\tau,y)|\, dtdx = 0. \tag{3.2}$$

We can now prove the following

3.7. **Lemma.** *Let $(\eta, \mathbf{q}) \in \mathcal{P}$ be any convex entropy-entropy pair and let $(\tilde{\tau}, \tilde{y}) \in \mathbb{R}^{d+1}$. Then for $\mathcal{H}^d$-almost every $(\tau, y) \in \partial\Pi^{\tilde{\tau},\tilde{y}}$:*

$$\lim_{r\to 0} \frac{1}{r^{d+1}} \int_{B_r^+(\tau,y)} |\mathbf{q}(u(t,x)) - \overline{\mathbf{q}(u)}_r(\tau,y)| dtdx = 0, \tag{3.3}$$

where $\overline{\mathbf{q}(u)}_r(\tau, y)$ is the average of $\mathbf{q}(u)$ over the half ball $B_r^+(\tau, y)$.

Proof. We fix $(\tau, y) \in \Pi^{\tilde{\tau},\tilde{y}}$ for which (3.2) holds. Given any $r_k \to 0$ there exists a subsequence (denoted again as r_k) and a constant $u_\infty^{\tau,y}$ (that depends on r_k) such that ([9]):

$$u_{r_k}^{\tau,y} \to u_\infty^{\tau,y} \text{ in } L^1_{loc}(\Pi^{\tau,y}), \tag{3.4}$$

where

$$u_{r_k}^{\tau,y}(t,x) := u(\tau + r_k t, y + r_k x), \quad (t,x) \in \Pi^{\tau,y}.$$

Therefore, for a further subsequence,

$$u_{r_k}^{\tau,y}(t,x) \to u_\infty^{\tau,y} \text{ for } \mathcal{L}^{d+1}\text{-a.e. } (t,x) \in B_1^+(\mathbf{0}),$$

which yields, for any $(\eta, \mathbf{q}) \in \mathcal{P}$, since $\mathbf{q}$ is continuous:

$$\mathbf{q}(u_{r_k}^{\tau,y}(t,x)) \to \mathbf{q}(u_\infty^{\tau,y}) \text{ for } \mathcal{L}^{d+1}\text{-a.e. } (t,x) \in B_1^+(\mathbf{0}). \tag{3.5}$$

From (3.5), the dominated convergence theorem (recall that u is bounded) and performing the change of variables $\alpha = \tau + r_k t$, $\xi = y + r_k x$ we obtain:

$$\frac{1}{r_k^{d+1}} \int_{B_{r_k}^+(\tau,y)} |\mathbf{q}(u(\alpha,\xi)) - \mathbf{q}(u_\infty^{\tau,y})| d\alpha d\xi \to 0 \text{ as } r_k \to 0. \tag{3.6}$$

For simplicity we write $u_\infty^{\tau,y} := u_\infty$, and $\overline{\mathbf{q}(u)}_r(\tau,y) := \overline{\mathbf{q}(u)}_r$ in the rest of the proof. We compute, for any $r > 0$,

$$\begin{aligned} \frac{1}{r^{d+1}} \int_{B_r^+(\tau,y)} |\mathbf{q}(u(t,x)) - \overline{\mathbf{q}(u)}_r| dtdx &\leq \frac{1}{r^{d+1}} \int_{B_r^+(\tau,y)} |\mathbf{q}(u(t,x)) - \mathbf{q}(u_\infty)| dtdx \\ &+ \frac{1}{r^{d+1}} \int_{B_r^+(\tau,y)} |\overline{\mathbf{q}(u)}_r - \mathbf{q}(u_\infty)| dtdx. \end{aligned} \tag{3.7}$$

On the other hand, if $\omega(d+1)$ denotes the $\mathcal{L}^{d+1}$-measure of the unit ball in $\mathbb{R}^{d+1}$ we have

$$\begin{aligned} \frac{1}{r^{d+1}} \int_{B_r^+(\tau,y)} |\overline{\mathbf{q}(u)}_r - \mathbf{q}(u_\infty)| dtdx &= \frac{\omega(d+1)}{2} |\overline{\mathbf{q}(u)}_r - \mathbf{q}(u_\infty)| \\ &= \frac{1}{r^{d+1}} \left| \int_{B_r^+(\tau,y)} \mathbf{q}(u(t,x)) - \mathbf{q}(u_\infty) dtdx \right| \\ &\leq \frac{1}{r^{d+1}} \int_{B_r^+(\tau,y)} |\mathbf{q}(u(t,x)) - \mathbf{q}(u_\infty)| dtdx \end{aligned} \tag{3.8}$$

From (3.7) and (3.8) we conclude

$$\frac{1}{r^{d+1}} \int_{B_r^+(\tau,y)} |\mathbf{q}(u(t,x)) - \overline{\mathbf{q}(u)}_r| dtdx \leq 2 \frac{1}{r^{d+1}} \int_{B_r^+(\tau,y)} |\mathbf{q}(u(t,x)) - \mathbf{q}(u_\infty)| dtdx,$$

and therefore (3.6) yields

$$\frac{1}{r_k^{d+1}} \int_{B_{r_k}^+(\tau,y)} |\mathbf{q}(u(t,x)) - \overline{\mathbf{q}(u)}_{r_k}| dtdx \to 0 \text{ as } r_k \to 0, \tag{3.9}$$

which implies (3.3). The dependence of (3.3) on the subsequence is illusory. The reason is that, if there where a subsequence $r_k \to 0$ such that

$$\frac{1}{r_k^{d+1}} \int_{B_{r_k}^+(\tau,y)} |\mathbf{q}(u(t,x)) - \overline{\mathbf{q}(u)}_{r_k}| dtdx \to l \neq 0,$$

then one could appeal to the previous argument to conclude that, for some further subsequence,

$$\frac{1}{r_k^{d+1}} \int_{B^+_{r_k}(\tau,y)} |\mathbf{q}(u(t,x)) - \overline{\mathbf{q}(u)}_{r_k}| dtdx \to 0,$$

which is contrary to our assertion that $l \neq 0$. □

We will need the following result (see Giusti [12], Lemma 2.3):

3.8. **Lemma.** *Let μ be a positive Radon measure in $\mathbb{R}_+ \times \mathbb{R}^d$. Then, for $\mathcal{H}^d$-almost every $y \in \mathbb{R}^d$,*

$$\lim_{r \to 0} \frac{\mu(C_r^+(y))}{r^d} = 0.$$

Assuming (3.1), the following theorem shows that $\eta(\mathbf{u})$ has strong traces $\mathcal{H}^d$-almost everywhere on hyperplanes.

3.9. **Theorem.** *Let $(\eta, \mathbf{q})$ be any convex entropy-entropy pair and let $(\tilde{s}, \tilde{y}) \in \mathbb{R}^{d+1}$. If $\mathbf{u} := (u^1, u^2, ..., u^m) \in L^\infty(\mathbb{R}^{d+1}; \mathbb{R}^m)$ satisfies the entropy inequality (2.8) and condition (3.1) on $\partial \Pi^{\tilde{s},\tilde{y}}$ then $\eta(\mathbf{u})$ has a strong trace at $\partial \Pi^{\tilde{s},\tilde{y}}$; that is, there exists a function $\eta(\mathbf{u})_{tr} \in L^\infty(\partial \Pi^{\tilde{s},\tilde{y}})$ such that, for $\mathcal{H}^d$-almost every $(s,y) \in \partial \Pi^{\tilde{s},\tilde{y}}$,*

$$\lim_{r \to 0} \frac{1}{r^{d+1}} \int_{B_r^+(s,y)} \eta(\mathbf{u}(t,x))\, dtdx = \eta(\mathbf{u})_{tr}(s,y). \tag{3.10}$$

In particular, if we choose $\eta = u^i$, $i = 1, ..., m$, we obtain the strong trace for each component of $\mathbf{u}$.

Proof. **Step 1:** We will apply the Gauss-Green formula given by Theorem 2.11 to

$$\boldsymbol{F}_{\mathbf{u}}^{\eta,\mathbf{q}}(t,x) := (\eta(\mathbf{u}(t,x)), \mathbf{q}(\mathbf{u}(t,x))),$$

which is a divergence-measure field. Indeed, as explained in §2.1, we have that

$$\boldsymbol{F}_{\mathbf{u}}^{\eta,\mathbf{q}} \in \mathcal{DM}^\infty_{loc}(\mathbb{R}^N). \tag{3.11}$$

Without loss of generality, and to simplify the exposition, we will prove (3.10) for the hyperplane $\Pi := \{t = 0\}$. Theorem 2.11 gives the existence of a function $\mathcal{F} \cdot \boldsymbol{\nu} \in L^\infty(\Pi)$ which is the weak normal trace of the vector field $\boldsymbol{F}_{\mathbf{u}}^{\eta,\mathbf{q}}$ on $\partial\Pi$. Let $\mathcal{G} \subset \Pi$ be the set of all Lebesgue points of $\mathcal{F} \cdot \boldsymbol{\nu}$ for which Lemma 3.8 and property (3.1) holds. We have $\mathcal{H}^d(\Pi \setminus \mathcal{G}) = 0$. For the rest of the proof we identify $(0,y) \in \Pi$ with y. Also, to simplify our exposition and without loss of generality we can assume $\mathbf{0} \in \mathcal{G}$. We define

$$C_r^+ := C_r^+(\mathbf{0}) = \mathcal{B}_r(\mathbf{0}) \times (0,r). \tag{3.12}$$

From Theorem 2.11 we have

$$\int_{C_r^+} \mathrm{div}_{t,x}(\Phi \boldsymbol{F}_{\mathbf{u}}^{\eta,\mathbf{q}}(t,x)) dtdx = -\int_{\partial C_r^+} \Phi \mathcal{F} \cdot \boldsymbol{\nu} d\mathcal{H}^d, \tag{3.13}$$

for any $\Phi \in C_0^1(\mathbb{R}^{d+1})$. The following function

$$\Phi_r(t,x) = \varphi\left(\frac{x}{r}\right)(r-t),\ (t,x) \in \mathbb{R}^{d+1}, 0 \le \varphi \le 1,\ \mathrm{supp}\ \varphi \subset \mathcal{B}_1(\mathbf{0}),\ \int_{\mathbb{R}^d} \varphi(x) dx = 1,$$

will be used as a test function in (3.13). We recall that the point $(0, x)$ has been identified with x and, for simplicity of notation, Φ_r will be denoted simply as Φ. After substituting Φ, the right hand side of (3.13) becomes

$$\int_{C_r^+} \operatorname{div}_{t,x}(\Phi \boldsymbol{F}_{\mathbf{u}}^{\eta,\mathbf{q}}(t,x))dtdx = -\int_{\mathcal{B}_r(\mathbf{0})} r\varphi\left(\frac{x}{r}\right)\mathcal{F}\cdot\boldsymbol{\nu}(x)d\mathcal{H}^d(x).$$

The product rule for divergence-measure fields (2.6) yields

$$div\,(\Phi \boldsymbol{F}_{\mathbf{u}}^{\eta,\mathbf{q}}) = \boldsymbol{F}_{\mathbf{u}}^{\eta,\mathbf{q}}\cdot\nabla\Phi + \Phi div\,\boldsymbol{F}_{\mathbf{u}}^{\eta,\mathbf{q}}$$

and hence (using the notation $\mu_{\eta,\mathbf{q}} := div\,\boldsymbol{F}_{\mathbf{u}}^{\eta,\mathbf{q}}$):

$$\int_{C_r^+}\Phi d\mu_{\eta,\mathbf{q}} + \int_{C_r^+}\boldsymbol{F}_{\mathbf{u}}^{\eta,\mathbf{q}}\cdot\nabla\Phi dtdx = -\int_{\mathcal{B}_r(\mathbf{0})} r\varphi\left(\frac{x}{r}\right)\mathcal{F}\cdot\boldsymbol{\nu}(x)d\mathcal{H}^d.$$

Therefore,

$$\int_{C_r^+}\Phi d\mu_{\eta,\mathbf{q}} + \int_{C_r^+}(\eta(\mathbf{u})\Phi_t + \mathbf{q}(\mathbf{u})\cdot\nabla_x\Phi)dtdx = -\int_{\mathcal{B}_r(\mathbf{0})} r\varphi\left(\frac{x}{r}\right)\mathcal{F}\cdot\boldsymbol{\nu}(x)d\mathcal{H}^d,$$

and hence

$$\frac{1}{r^{d+1}}\int_{C_r^+}\Phi d\mu_{\eta,\mathbf{q}} + \frac{1}{r^{d+1}}\int_{C_r^+}(\eta(\mathbf{u})\Phi_t + \mathbf{q}(\mathbf{u})\cdot\nabla_x\Phi)dtdx = -\frac{1}{r^{d+1}}\int_{\mathcal{B}_r(\mathbf{0})} r\varphi\left(\frac{x}{r}\right)\mathcal{F}\cdot\boldsymbol{\nu}(x)d\mathcal{H}^d.$$

Using the definition of Φ we obtain

$$\frac{1}{r^{d+1}}\int_{C_r^+}\eta(\mathbf{u})\Phi_t = -\frac{1}{r^{d+1}}\int_{C_r^+}\varphi\left(\frac{x}{r}\right)\eta(\mathbf{u})(t,x)dtdx.$$

Also, since

$$-\frac{1}{r^{d+1}}\int_{\mathcal{B}_r(\mathbf{0})} r\varphi\left(\frac{x}{r}\right)\mathcal{F}\cdot\boldsymbol{\nu}d\mathcal{H}^d = -\frac{1}{r^d}\int_{\mathcal{B}_r(\mathbf{0})}\varphi\left(\frac{x}{r}\right)\mathcal{F}\cdot\boldsymbol{\nu}(x)d\mathcal{H}^d,$$

the following equation is obtained:

$$\begin{aligned}\frac{1}{r^{d+1}}\int_{C_r^+}(r-t)\varphi\left(\frac{x}{r}\right)d\mu_{\eta,\mathbf{q}} \quad &- \quad \frac{1}{r^{d+1}}\int_{C_r^+}\varphi\left(\frac{x}{r}\right)\eta(\mathbf{u}(t,x))dtdx \\ &+ \quad \frac{1}{r^{d+1}}\int_{C_r^+}\mathbf{q}(\mathbf{u})\cdot\nabla_x\Phi dtdx = -\frac{1}{r^d}\int_{\mathcal{B}_r(\mathbf{0})}\varphi\left(\frac{x}{r}\right)\mathcal{F}\cdot\boldsymbol{\nu}(x)d\mathcal{H}^d.\end{aligned} \tag{3.14}$$

Step 2: We will now show in (3.14) that, as $r \to 0$,

$$\frac{1}{r^{d+1}}\int_{C_r^+}\mathbf{q}(\mathbf{u})\cdot\nabla_x\Phi dtdx \to 0 \tag{3.15}$$

and

$$\frac{1}{r^{d+1}}\int_{C_r^+}(r-t)\varphi\left(\frac{x}{r}\right)d\mu_{\eta,\mathbf{q}} \to 0. \tag{3.16}$$

We have

$$
\begin{aligned}
\frac{1}{r^{d+1}}\int_{\mathcal{B}_r(\mathbf{0})\times(0,r)} \mathbf{q}(\mathbf{u})\cdot\nabla_x\Phi dtdx &= \\
&= \frac{1}{r}\cdot\frac{1}{r^{d+1}}\int_{\mathcal{B}_r(\mathbf{0})\times(0,r)}(r-t)\mathbf{q}(\mathbf{u}(t,x))\cdot\nabla_x\varphi\left(\frac{x}{r}\right)dtdx \\
&= \frac{1}{r}\int_{\mathcal{B}_1(\mathbf{0})\times(0,1)}(r-r\tau)\mathbf{q}(\mathbf{u}(r\tau,r\xi))\cdot\nabla_x\varphi(\xi)d\tau d\xi \\
&= \int_{\mathcal{B}_1(\mathbf{0})\times(0,1)}(1-\tau)\mathbf{q}(\mathbf{u}(r\tau,r\xi))\cdot\nabla_x\varphi(\xi)d\tau d\xi,
\end{aligned}
$$

where the following change of variables has been performed: $t=r\tau$ and $x=r\xi$. If $\overline{\mathbf{q}(\mathbf{u})}_r$ denotes the average of $\mathbf{q}(\mathbf{u})$ in the cylinder $\mathcal{B}_r(\mathbf{0})\times(0,r)$ then

$$
\overline{\mathbf{q}(\mathbf{u})}_r\cdot\int_0^1(1-\tau)\left[\int_{\mathcal{B}_1(\mathbf{0})}\nabla_x\varphi(\xi)d\xi\right]d\tau=0,
$$

since φ has compact support in $\mathcal{B}_1(\mathbf{0})$. Therefore, with $C_1^+=\mathcal{B}_1(\mathbf{0})\times(0,1)$ and using (3.1) we compute

$$
\begin{aligned}
\left|\frac{1}{r^{d+1}}\int_{\mathcal{B}_r(\mathbf{0})\times(0,r)}\mathbf{q}(\mathbf{u})\cdot\nabla_x\Phi dtdx\right| &= |\int_{C_1^+}(1-\tau)\mathbf{q}(\mathbf{u}(r\tau,r\xi))\cdot\nabla_x\varphi(\xi)d\tau d\xi \\
&- \overline{\mathbf{q}(\mathbf{u})}_r\cdot\int_{C_1^+}(1-\tau)\nabla_x\varphi(\xi)d\tau d\xi| \\
&= \left|\int_{C_1^+}(1-\tau)(\mathbf{q}(\mathbf{u}(r\tau,r\xi))-\overline{\mathbf{q}(\mathbf{u})}_r)\cdot\nabla_x\varphi(\xi)d\tau d\xi\right| \\
&\le \int_{C_1^+}|\mathbf{q}(\mathbf{u}(r\tau,r\xi))-\overline{\mathbf{q}(\mathbf{u})}_r||\nabla_x\varphi(\xi)|d\tau d\xi \\
&\le C\int_{C_1^+}|\mathbf{q}(\mathbf{u}(r\tau,r\xi))-\overline{\mathbf{q}(\mathbf{u})}_r|d\tau d\xi \\
&= C\frac{1}{r^{d+1}}\int_{C_r^+}|\mathbf{q}(\mathbf{u}(t,x))-\overline{\mathbf{q}(\mathbf{u})}_r|dtdx \\
&\to 0 \text{ as } r\to 0,
\end{aligned}
$$

which proves (3.15). We now proceed to show (3.16). Using that $0\le\varphi\le 1$ we compute

$$
\left|\frac{1}{r^{d+1}}\int_{C_r^+}(r-t)\varphi\left(\frac{x}{r}\right)d\mu_{\eta,\mathbf{q}}\right|\le\frac{1}{r^{d+1}}\int_{C_r^+}r\left|\varphi\left(\frac{x}{r}\right)\right|d\mu_{\eta,\mathbf{q}}\le\frac{1}{r^d}\|\mu_{\eta,\mathbf{q}}\|(C_r^+)\to 0
$$

due to Lemma 3.8.

Step 3: From (3.14), (3.15), (3.16) we obtain, for any $\varphi\in C_0^\infty(\mathcal{B}_1(\mathbf{0}))$,

$$
\lim_{r\to 0}\frac{1}{r^{d+1}}\int_{C_r^+}\varphi\left(\frac{x}{r}\right)\eta(\mathbf{u}(t,x))dtdx=\lim_{r\to 0}\frac{1}{r^{d+1}}\int_{\mathcal{B}_r(\mathbf{0})}\varphi\left(\frac{x}{r}\right)\mathcal{F}\cdot\boldsymbol{\nu}(x)d\mathcal{H}^d, \tag{3.17}
$$

and performing the change of variables $t = r\tau$ and $x = r\xi$ we obtain, for any $\varphi \in C_0^\infty(\mathcal{B}_1(\mathbf{0}))$,

$$\lim_{r\to 0}\int_{C_1^+}\varphi(\xi)\eta(\mathbf{u}(r\tau, r\xi))d\tau d\xi = \lim_{r\to 0}\int_{\mathcal{B}_1(\mathbf{0})}\varphi(\xi)\mathcal{F}\cdot\boldsymbol{\nu}(r\xi)d\mathcal{H}^d. \tag{3.18}$$

Both limits in (3.18) exist because $\mathbf{0}$ is a Lebesgue point of the normal trace function $\mathcal{F}\cdot\boldsymbol{\nu}$. Since (3.18) holds for any test function φ with compact support in $\mathcal{B}_1(\mathbf{0})$, we can choose a sequence $\varphi_k \in C_0^\infty(\mathcal{B}_1(\mathbf{0}))$ such that $\varphi_k \to 1$ pointwise in $\mathcal{B}_1(\mathbf{0})$. Therefore, the following limits exist, for each k,

$$\lim_{r\to 0}\int_{C_1^+}\varphi_k(\xi)\eta(\mathbf{u}(r\tau, r\xi))d\tau d\xi = \lim_{r\to 0}\int_{\mathcal{B}_1(\mathbf{0})}\varphi_k(\xi)\mathcal{F}\cdot\boldsymbol{\nu}(r\xi)d\mathcal{H}^d. \tag{3.19}$$

We define

$$h_k(r) := \int_{C_1^+}\varphi_k(\xi)\eta(\mathbf{u}(r\tau, r\xi))d\tau d\xi,$$

and we note that $h_k \to h$ uniformly on r, where

$$h(r) := \int_{C_1^+}\eta(\mathbf{u}(r\tau, r\xi))d\tau d\xi.$$

Also, from (3.19) the following limit exists

$$A_k := \lim_{r\to 0}\int_{C_1^+}\varphi_k(\xi)\eta(\mathbf{u}(r\tau, r\xi))d\tau d\xi.$$

Therefore we conclude that (see, for example, Rudin [16], Theorem 7.11):

$$\lim_{k\to\infty} A_k \text{ exists}$$

and

$$\lim_{r\to 0}\lim_{k\to\infty} h_k(r) = \lim_{k\to\infty}\lim_{r\to 0} h_k(r). \tag{3.20}$$

Proceeding in the same way with the right hand side of (3.19) we conclude:

$$\lim_{k\to\infty}\lim_{r\to 0}\int_{\mathcal{B}_1(\mathbf{0})}\varphi_k(\xi)\mathcal{F}\cdot\boldsymbol{\nu}(r\xi)d\mathcal{H}^d = \lim_{r\to 0}\lim_{k\to\infty}\int_{\mathcal{B}_1(\mathbf{0})}\varphi_k(\xi)\mathcal{F}\cdot\boldsymbol{\nu}(r\xi)d\mathcal{H}^d,$$

and therefore

$$\begin{aligned}
\lim_{r\to 0}\int_{C_1^+}\eta(\mathbf{u}(r\tau,r\xi))d\tau d\xi &= \lim_{r\to 0}\lim_{k\to\infty}\int_{C_1^+}\varphi_k(\xi)\eta(\mathbf{u}(r\tau,r\xi))d\tau d\xi\\
&= \lim_{k\to\infty}\lim_{r\to 0}\int_{C_1^+}\varphi_k(\xi)\eta(\mathbf{u}(r\tau,r\xi))d\tau d\xi\\
&= \lim_{k\to\infty}\lim_{r\to 0}\int_{\mathcal{B}_1(\mathbf{0})}\varphi_k(\xi)\mathcal{F}\cdot\boldsymbol{\nu}(r\xi)d\mathcal{H}^d\\
&= \lim_{r\to 0}\lim_{k\to\infty}\int_{\mathcal{B}_1(\mathbf{0})}\varphi_k(\xi)\mathcal{F}\cdot\boldsymbol{\nu}(r\xi)d\mathcal{H}^d\\
&= \lim_{r\to 0}\int_{\mathcal{B}_1(\mathbf{0})}\mathcal{F}\cdot\boldsymbol{\nu}(r\xi)d\mathcal{H}^d.
\end{aligned}$$

Hence,

$$\lim_{r\to 0}\int_{C_1^+}\eta(\mathbf{u}(r\tau,r\xi))d\tau d\xi = \lim_{r\to 0}\int_{\mathcal{B}_1(\mathbf{0})}\mathcal{F}\cdot\boldsymbol{\nu}(r\xi)d\mathcal{H}^d. \tag{3.21}$$

Changing variables back in (3.21) yields

$$\lim_{r\to 0}\frac{1}{r^{d+1}}\int_{\mathcal{B}_r(\mathbf{0})\times(0,r)}\eta(\mathbf{u}(t,x))dtdx = \lim_{r\to 0}\frac{1}{r^d}\int_{\mathcal{B}_r(\mathbf{0})}\mathcal{F}\cdot\boldsymbol{\nu}(x)d\mathcal{H}^d,$$

and, since $\mathbf{0}$ is a Lebesgue point of $\mathcal{F}\cdot\boldsymbol{\nu}$, we conclude

$$\lim_{r\to 0}\frac{1}{r^{d+1}}\int_{C_r^+}\eta(\mathbf{u}(t,x))dtdx = \mathcal{F}\cdot\boldsymbol{\nu}(\mathbf{0}).$$

□

4. Strong traces on sets of finite perimeter

We first introduce the following definitions:

4.1. **Definition.** If E is a set of finite perimeter, we define for $\mathcal{H}^d$-almost every $\mathbf{z}\in\partial^*E$,

$$\mathbf{T}(\mathbf{z}) := \{\mathbf{x} : (\mathbf{x}-\mathbf{z})\cdot\boldsymbol{\nu}(\mathbf{z}) = 0\},$$

where $\boldsymbol{\nu}(\mathbf{z})$ is the inner unit normal at $\mathbf{z}$. We also define the cylinder

$$C_r^+(\mathbf{z}) := (B_r(\mathbf{z})\cap\mathbf{T}(\mathbf{z}))\times(0,r),$$

and

$$C_r(\mathbf{z}) := (B_r(\mathbf{z})\cap\mathbf{T}(\mathbf{z}))\times(-r,r).$$

4.2. *Remark.* Since the ball $B_r(\mathbf{z})$ can be inscribed in the cylinder $C_r(\mathbf{z})$, then the results in this section can be stated with balls or cylinders.

4.3. **Definition.** Let $E \subset \mathbb{R}^{d+1}$ be a set of finite perimeter. We say that $\mathbf{u}$ satisfies the vanishing mean oscillation property on half balls if, for any continuous $\mathbf{q} \in C(\mathbb{R}^m, \mathbb{R}^d)$ and $\mathcal{H}^d$-almost every $\mathbf{z} \in \partial^* E$:

$$\lim_{r \to 0} \frac{1}{r^{d+1}} \int_{B_r^+(\mathbf{z})} |\mathbf{q}(\mathbf{u}(t,x)) - \overline{\mathbf{q}(\mathbf{u})}_r(\mathbf{z})| \, dtdx = 0, \tag{4.1}$$

where $\overline{q(\mathbf{u})}_r(\mathbf{z})$ is the vector in $\mathbb{R}^d$ which is the average of $\mathbf{q}(\mathbf{u})$ over the half ball $B_r^+(\mathbf{z}) := B_r(\mathbf{z}) \cap \{\mathbf{x} : (\mathbf{x} - \mathbf{z}) \cdot \boldsymbol{\nu}(\mathbf{z}) > 0\}$.

4.4. *Remark.* Condition (4.1) holds for the scalar case (i.e. $m = 1$) due to the rectifiability of the set of shock waves, proven in De-Lellis-Otto-Westdickenberg [9] (recall from Theorem 2.7 that $\partial^* E$ is also a d-rectifiable set).

We now proceed to extend Lemma 3.8 to the case of sets of finite perimeter:

4.5. **Lemma.** *Let E be a bounded set of finite perimeter and let μ be a positive Radon measure in $\mathbb{R}^{d+1}$ such that $\mu << \mathcal{H}^d$. Then, for $\mathcal{H}^d$-almost every $\mathbf{z} \in \partial^* E$,*

$$\lim_{r \to 0} \frac{\mu(D_r^{\mathbf{z}})}{r^d} = 0, \tag{4.2}$$

where $D_r^{\mathbf{z}} = E^1 \cap C_r(\mathbf{z})$.

Proof. Since $\mu << \mathcal{H}^d$, Corollary 4.4 in Chen-Torres-Ziemer [7] gives the existence of a sequence of smooth sets A_l such that

$$\lim_{l \to \infty} \mu(A_l \Delta E^1) = 0, \tag{4.3}$$

and

$$\lim_{l \to \infty} \mathcal{H}^d(\partial A_l \cap (\partial^* E \cup E^0)) = 0 \tag{4.4}$$

Therefore, given any $\varepsilon > 0$, there exists an smooth set A_ε such that

$$\mu(E^1 \setminus A_\varepsilon) < \varepsilon. \tag{4.5}$$

We let

$$S_k = \{\mathbf{z} \in \partial^* E : \limsup_{r \to 0} \frac{\mu(D_r^{\mathbf{z}})}{r^d} > \frac{1}{k}\}.$$

It is enough to show that $\mathcal{H}^d(S_k) = 0$ for each k. We let S_k^ε denote the set of all $\mathbf{z} \in S_k$ such that there exists $r_\mathbf{z}$ so that

$$D_{r_\mathbf{z}}^{\mathbf{z}} \subset E^1 \setminus A_\varepsilon.$$

Due to (4.3) and (4.4) we have that $S_k^{\varepsilon_i}$, with $\varepsilon_i \to 0$, is an increasing sequence of sets and $S_k = \cup S_k^{\varepsilon_i}$. Hence $\mathcal{H}^d(S_k) = \lim_{i \to \infty} \mathcal{H}^d(S_k^{\varepsilon_i})$. We now proceed to show that $\mathcal{H}^d(S_k^{\varepsilon_i}) < c(N)\varepsilon_i$, where $c(N)$ is a constant that depends on dimension $N = d + 1$. For each $\mathbf{z} \in S_k^{\varepsilon_i}$ (choosing smaller $r_\mathbf{z}$ if necessary) the definition of S_k implies

$$\frac{\mu(D_{r_\mathbf{z}}^{\mathbf{z}})}{r^d} > \frac{1}{2k}. \tag{4.6}$$

By choosing even smaller $r_{\mathbf{z}}$ if necessary we can also assume (see Giusti [12], Lemma 3.5, page 45):

$$\mathcal{H}^d(\partial^* E \cap C_r(\mathbf{z})) \leq c(N) r_{\mathbf{z}}^d. \tag{4.7}$$

A covering argument yields a countable sequence $\mathbf{z}_j \subset S_k^\varepsilon$ such that the sets $F_j := \partial^* E \cap C_{r_j}(\mathbf{z}_j), r_j := r_{\mathbf{z}_j}$, are pairwise disjoints and $S_k^\varepsilon \subset \cup G_j$ where $F_j \subset G_j$ and $\mathcal{H}^d(G_j) \leq c(N)\mathcal{H}^d(F_j)$. Thus, from (4.5), (4.6) and (4.7) we can estimate $\mathcal{H}^d(S_k^{\varepsilon_i})$ in terms of $\mu(E^1 \setminus A_{\varepsilon_i})$:

$$\begin{aligned} \mathcal{H}^d(S_k^{\varepsilon_i}) &\leq c(N) \sum \mathcal{H}^d(F_j) \\ &= c(N) \sum \mathcal{H}^d(\partial^* E \cap C_{r_j}(\mathbf{z}_j)) \\ &\leq c(N) \sum r_j^d < 2kc(N) \sum \mu(D_{r_j}^{\mathbf{z}_j}) \\ &\leq c(N)\mu(E^1 \setminus A_\varepsilon) < c(N)\varepsilon_i, \end{aligned}$$

which implies $\mathcal{H}^d(S_k) = \lim_{\varepsilon_i \to 0} \mathcal{H}^d(S_k^{\varepsilon_i}) \leq \lim_{\varepsilon_i \to 0} c(N)\varepsilon_i = 0$. □

Given any bounded set of finite perimeter $E \subset \mathbb{R}^{d+1}$, Theorem 2.11 yields the existence of the weak interior normal trace $\mathcal{F} \cdot \boldsymbol{\nu} \in L^\infty(\partial^* E)$ of the vector field

$$\boldsymbol{F}_{\mathbf{u}}^{\eta,\mathbf{q}}(t,x) := (\eta(\mathbf{u}(t,x)), \mathbf{q}(\mathbf{u}(t,x)))$$

on $\partial^* E$. We note that $\mathcal{H}^d$-almost every $\mathbf{z} \in \partial^* E$ is a Lebesgue point of $\mathcal{F} \cdot \boldsymbol{\nu}$. Assuming (4.1) we show next that the weak trace of the vector field $\boldsymbol{F}_{\mathbf{u}}^{\eta,\mathbf{q}}$ is indeed strong:

4.6. **Theorem.** *Let $E \subset \mathbb{R}^{d+1}$ be a bounded set of finite perimeter. If $\mathbf{u} \in L^\infty(\mathbb{R}^{d+1}, \mathbb{R}^m)$ satisfies the entropy inequality* (2.8) *and property* (4.1) *then there exists a function $\mathcal{F} \cdot \boldsymbol{\nu} \in L^\infty(\partial^* E)$ such that, for every convex entropy-entropy pair $(\eta, \mathbf{q}) \in \mathcal{P}$ and for $\mathcal{H}^d$-almost every $\mathbf{z} \in \partial^* E$,*

$$\lim_{r \to 0} \frac{1}{r^{d+1}} \int_{E^1 \cap B_r(\mathbf{z})} (\eta(\mathbf{u}(t,x)), \mathbf{q}(\mathbf{u}(t,x))) \cdot \boldsymbol{\nu}(\mathbf{z}) dt dx = \mathcal{F} \cdot \boldsymbol{\nu}(\mathbf{z}). \tag{4.8}$$

Proof. Let $\mathcal{F} \cdot \boldsymbol{\nu}$ denote the weak interior normal trace of the vector field

$$\boldsymbol{F}_{\mathbf{u}}^{\eta,\mathbf{q}} := (\eta(\mathbf{u}), \mathbf{q}(\mathbf{u}))$$

on $\partial^* E$ given by Theorem 2.11. Let $\mathcal{G} \subset \partial^* E$ be the set of all Lebesgue points of $\mathcal{F} \cdot \boldsymbol{\nu}$ that satisfy Lemma 4.5 and property (4.1). We have that $\mathcal{H}^d(\partial^* E \setminus \mathcal{G}) = 0$.

We split the proof in two steps:

Step 1: We first consider the case $\mathbf{0} \in \mathcal{G}$ and $\boldsymbol{\nu}(\mathbf{0}) = (1, 0, ..., 0)$. Proceeding as in §3 we can show

$$\begin{aligned} \lim_{r \to 0} \frac{1}{r^{d+1}} \int_{E^1 \cap C_r(\mathbf{0})} \eta(\mathbf{u}(t,x)) dt dx &= \lim_{r \to 0} \frac{1}{r^{d+1}} \int_{E^1 \cap C_r(\mathbf{0})} (\eta(\mathbf{u}(t,x)), \mathbf{q}(\mathbf{u}(t,x))) \cdot \boldsymbol{\nu}(\mathbf{0}) dt dx \\ &= \mathcal{F} \cdot \boldsymbol{\nu}(\mathbf{0}). \end{aligned} \tag{4.9}$$

Indeed, if we appy Theorem 2.11 to

$$D_r := E \cap C_r,$$

where (to simplify notation):

$$C_r := C_r(\mathbf{0}) \tag{4.10}$$

we obtain

$$\int_{(D_r)^1} \mathrm{div}_{t,x}(\Phi \boldsymbol{F}_{\mathbf{u}}^{\eta,\mathbf{q}}(t,x))dtdx = -\int_{\partial^* D_r} \Phi \mathcal{F}\cdot \boldsymbol{\nu} d\mathcal{H}^d, \tag{4.11}$$

for any $\Phi \in C_0^1(\mathbb{R}^{d+1})$. Choosing Φ as in §3 and noting that

$$\mathcal{H}^d([\partial^* D_r]\Delta[(\partial^* E\cap C_r)\cup(\partial^* C_r \cap E)]) = 0$$

and

$$(D_r)^1 = E^1 \cap C_r,$$

(4.11) becomes:

$$\int_{E^1\cap C_r} \mathrm{div}_{t,x}(\Phi \boldsymbol{F}_{\mathbf{u}}^{\eta,\mathbf{q}}(t,x))dtdx = -\int_{\partial^* E\cap C_r} \Phi \mathcal{F}\cdot \boldsymbol{\nu} d\mathcal{H}^d. \tag{4.12}$$

From (4.12) and proceeding as in §3 we obtain

$$\begin{aligned}\frac{1}{r^{d+1}}\int_{E^1\cap C_r}(r-t)\varphi\left(\frac{x}{r}\right)d\mu_{\eta,\mathbf{q}} \quad &- \quad \frac{1}{r^{d+1}}\int_{E^1\cap C_r}\varphi\left(\frac{x}{r}\right)\eta(\mathbf{u}(t,x))dtdx\\ &+ \quad \frac{1}{r^{d+1}}\int_{E^1\cap C_r}\mathbf{q}(\mathbf{u})\cdot\nabla_x\Phi dtdx\\ &= \quad -\frac{1}{r^d}\int_{\partial^* E\cap C_r}\varphi\left(\frac{x}{r}\right)\mathcal{F}\cdot\boldsymbol{\nu}(t,x)d\mathcal{H}^d.\end{aligned} \tag{4.13}$$

The difference between this case and the one considered in §3 is that $\partial^* E$ is not flat, but this can be overcome by using the regularity of the reduced boundary in Theorem 2.7. In particular, Theorem 2.7 states that

$$\frac{|(E^1\cap C_r)\Delta C_r^+|}{r^{d+1}} \to 0, \text{ as } r\to 0,$$

and therefore, using property (4.1) and proceeeding as in §3 we obtain

$$\frac{1}{r^{d+1}}\int_{E^1\cap C_r}\mathbf{q}(\mathbf{u})\cdot\nabla_x\Phi dtdx \to 0 \text{ as } r\to 0. \tag{4.14}$$

Also, since $\mu_{\eta,\mathbf{q}} << \mathcal{H}^d$ (see Remark 2.13), Lemma 4.5 implies

$$\frac{1}{r^{d+1}}\int_{E^1\cap C_r}(r-t)\varphi\left(\frac{x}{r}\right)d\mu_{\eta,\mathbf{q}} \le 2\frac{\|\mu_{\eta,\mathbf{q}}\|(E^1\cap C_r)}{r^d} \to 0, \text{ as } r\to 0. \tag{4.15}$$

Therefore, (4.13) reduces to

(4.16)
$$\lim_{r\to 0}\frac{1}{r^{d+1}}\int_{E^1\cap C_r}\varphi\left(\frac{x}{r}\right)\eta(\mathbf{u}(t,x))dtdx = \lim_{r\to 0}\frac{1}{r^{d+1}}\int_{\partial^* E\cap C_r}\varphi\left(\frac{x}{r}\right)\mathcal{F}\cdot\boldsymbol{\nu}(t,x)d\mathcal{H}^d,$$

for any $\varphi \in C_0^\infty(\mathcal{B}_1(\mathbf{0}))$. If we proceed now as in §3 we obtain

$$\lim_{r\to 0}\frac{1}{r^{d+1}}\int_{E^1\cap C_r}\eta(\mathbf{u}(t,x))dtdx = \lim_{r\to 0}\frac{1}{r^d}\int_{\partial^* E\cap C_r}\mathcal{F}\cdot\boldsymbol{\nu} d\mathcal{H}^d,$$

and, since $\mathbf{0}$ is a Lebesgue point of $\mathcal{F}\cdot\boldsymbol{\nu}$, we conclude

$$\lim_{r\to 0}\frac{1}{r^{d+1}}\int_{E^1\cap C_r}\eta(\mathbf{u}(t,x))dtdx=\mathcal{F}\cdot\boldsymbol{\nu}(\mathbf{0}).$$

Step 2: We now fix any $\mathbf{z}\in\mathcal{G}$.

We perform the change of variables

$$\mathbf{y}:=(y_0,y_1,...,y_N)=T(\mathbf{x}),\ \mathbf{x}=(t,x), \tag{4.17}$$

so that $T(\boldsymbol{\nu}(\mathbf{z}))=(1,0,...,0)$ (this change of variables was used in [18] and [19] in the scalar case) and, without loss of generality, we assume that $T(\mathbf{z})=\mathbf{0}$. The equation in the new coordinates is

$$div_{\mathbf{y}}\tilde{\mathbf{f}}(\tilde{\mathbf{u}}(\mathbf{y})))=0,$$

where

$$\tilde{\mathbf{u}}(\mathbf{y})=\mathbf{u}(\mathbf{x}),\quad \mathbf{x}=T^{-1}(\mathbf{y})$$

and (recall that $\mathbf{f}^i:\mathbb{R}^m\to\mathbb{R}^d$, $i=1,...,m$):

$$\tilde{\mathbf{f}}^i(\xi_1,...,\xi_m)=T(\xi_i,\mathbf{f}^i(\xi_1,...,\xi_m)),\quad i=1,...,m.$$

We also define, for any entropy pair $(\eta,\mathbf{q})$ (recall $\eta:\mathbb{R}^m\to\mathbb{R}$ and $\mathbf{q}:\mathbb{R}^m\to\mathbb{R}^d$):

$$\tilde{\mathbf{g}}_{\eta,\mathbf{q}}(\xi_1,...,\xi_m)=T(\eta(\xi_1,...,\xi_m),\mathbf{q}(\xi_i,...,\xi_m))$$

and

$$\tilde{C}_r=T(C_r(\mathbf{z})),\ \tilde{E}=T(E).$$

We define the vector field

$$\tilde{\boldsymbol{F}}^{\eta,\mathbf{q}}_{\mathbf{u}}(\mathbf{y}):=\tilde{g}_{\eta,\mathbf{q}}(\tilde{\mathbf{u}}(\mathbf{y})),$$

which is also a divergence-measure field in the new coordinates. In order to see this we note that, since T is a rotation, we have:

$$\begin{aligned}\int_{\mathbb{R}^N}\boldsymbol{F}^{\eta,\mathbf{q}}_{\mathbf{u}}(\mathbf{x})\cdot\nabla_{\mathbf{x}}\varphi(\mathbf{x})d\mathbf{x} &= \int_{\mathbb{R}^N}(\eta(\mathbf{u}(\mathbf{x})),\mathbf{q}(\mathbf{u}(\mathbf{x})))\cdot\nabla_{\mathbf{x}}\varphi(\mathbf{x})d\mathbf{x}\\ &= \int_{\mathbb{R}^N}T(\eta(\mathbf{u}(\mathbf{x})),\mathbf{q}(\mathbf{u}(\mathbf{x})))\cdot T\nabla_{\mathbf{x}}\varphi(\mathbf{x})d\mathbf{x}\\ &= \int_{\mathbb{R}^N}\tilde{g}^{\eta,\mathbf{q}}_{\mathbf{u}}(\tilde{\mathbf{u}}(\mathbf{y}))\cdot\nabla_{\mathbf{y}}\varphi(\mathbf{y})d\mathbf{y}\\ &= \int_{\mathbb{R}^N}\tilde{\boldsymbol{F}}^{\eta,\mathbf{q}}_{\mathbf{u}}(\mathbf{y})\cdot\nabla_{\mathbf{y}}\varphi(\mathbf{y})d\mathbf{y}\end{aligned}$$

for any $\varphi\in C_0^\infty(\mathbb{R}^N)$. Therefore, since $\boldsymbol{F}^{\eta,\mathbf{q}}_{\mathbf{u}}\in\mathcal{DM}^\infty_{loc}(\mathbb{R}^N)$ it follows that, for each bounded open set $D\subset\mathbb{R}^N$,

$$\sup\{\tilde{\boldsymbol{F}}^{\eta,\mathbf{q}}_{\mathbf{u}}\cdot\nabla_{\mathbf{y}}\varphi:\varphi\in C_0^\infty(D),|\varphi|\le 1,spt(\varphi)\subset D\}<\infty;$$

that is, $\tilde{\boldsymbol{F}}^{\eta,\mathbf{q}}_{\mathbf{u}}\in\mathcal{DM}^\infty_{loc}(\mathbb{R}^N)$. We denote the normal trace of the divergence-measure vector field $\tilde{\boldsymbol{F}}^{\eta,\mathbf{q}}_{\mathbf{u}}(\mathbf{y})$ on $\partial\tilde{E}$ as $\tilde{\mathcal{F}}\cdot\tilde{\boldsymbol{\nu}}$. Let $\tilde{\boldsymbol{\nu}}$ denote the normal to $\tilde{E}$. Since T is a rotation we have that $T(\mathbf{z})$ is a Lebesque point for $\tilde{\mathfrak{F}} \cdot \tilde{\boldsymbol{\nu}}$ and $\tilde{\mathbf{u}}$ satisfies (4.1) on $\tilde{E}$. Therefore, from Step 1 we obtain

$$\lim_{r\to 0} \frac{1}{r^{d+1}} \int_{\tilde{E}^1 \cap \tilde{C}_r} \tilde{\mathbf{g}}_{\eta,\mathbf{q}}(\tilde{\mathbf{u}}(\mathbf{y})) \cdot \tilde{\boldsymbol{\nu}}(\mathbf{0}) d\mathbf{y} = \tilde{\mathfrak{F}} \cdot \tilde{\boldsymbol{\nu}}(\mathbf{0}). \tag{4.18}$$

Since T is a rotation we have

$$\tilde{\mathbf{g}}_{\eta,\mathbf{q}}(\tilde{\mathbf{u}}(\mathbf{y})) \cdot \tilde{\boldsymbol{\nu}}(T(\mathbf{z})) = \mathbf{g}_{\eta,\mathbf{q}}(\mathbf{u}(\mathbf{x})) \cdot \boldsymbol{\nu}(\mathbf{z}), \quad \mathbf{x} = T^{-1}(\mathbf{y}) \tag{4.19}$$

and

$$\tilde{\mathfrak{F}} \cdot \tilde{\boldsymbol{\nu}}(T(\mathbf{z})) = \mathfrak{F} \cdot \boldsymbol{\nu}(\mathbf{z}). \tag{4.20}$$

Changing variables in (4.18) and using that $|\det T| = 1$ we conclude

$$\lim_{r\to 0} \frac{1}{r^{d+1}} \int_{E^1 \cap C_r(\mathbf{z})} \mathbf{g}_{\eta,\mathbf{q}}(\mathbf{u}(\mathbf{x})) \cdot \boldsymbol{\nu}(\mathbf{z}) d\mathbf{x} = \mathfrak{F} \cdot \boldsymbol{\nu}(\mathbf{z}),$$

which is our desired result:

$$\lim_{r\to 0} \frac{1}{r^{d+1}} \int_{E^1 \cap C_r(\mathbf{z})} (\eta(\mathbf{u}(t,x)), \mathbf{q}(\mathbf{u}(t,x))) \cdot \boldsymbol{\nu}(\mathbf{z}) dt dx = \mathfrak{F} \cdot \boldsymbol{\nu}(\mathbf{z}).$$

□

Acknowledgments. The author would like to especially thank Camillo De Lellis for many useful discussions. The author would like to thank also Alexis Vasseur and Constantine Dafermos for helpful comments. Monica Torres's research was supported in part by the National Science Foundation under grant DMS-0540869.

References

[1] L. Ambrosio, G. Crippa, and S. Maniglia. Traces and fine properties of a *BD* class of vector fields and applications. *Ann. Fac. Sci. Toulouse Math. (6)*, 14(4):527–561, 2005.

[2] L. Ambrosio, N. Fusco and D. Pallara. *Functions of bounded variation and free discontinuity problems.* Oxford mathematical monographs, 2000.

[3] G. Anzellotti. Pairings between measures and functions and compensated comptactness. *Ann. Mat. Pura Appl.*, 135:293–318, 1983.

[4] G.-Q. Chen and H. Frid. Divergence-measure fields and hyperbolic conservation laws. *Arch. Ration. Mech. Anal.*, 147(2):89–118, 1999.

[5] G.-Q. Chen and M. Rascle. Initial layers and uniqueness of weak entropy solutions to hyperbolic conservation laws. *Arch. Ration. Mech. Anal.*, 153(3):205:220, 2000.

[6] G.-Q. Chen and M. Torres. Divergence-measure fields, sets of finite perimeter, and conservation laws. *Arch. Ration. Mech. Anal.*, 175(2):245–267, 2005.

[7] G.-Q. Chen, M. Torres, and W. Ziemer. Gauss-Green theorem for weakly differentiable vector fields, sets of finite perimeter, and balance laws. *Communications on Pure and Applied Mathematics*, 2008 (to appear).

[8] G.-Q. Chen, M. Torres, and W. P. Ziemer. Measure-theoretical analysis and non-linear conservation laws. *Pure and Applied Mathematics Quarterly*, 3(3):841–879, 2007.

[9] C. De Lellis, F. Otto, and M. Westdickenberg. Structure of entropy solutions for multidimensional scalar conservation laws. *Arch. Rational Mech. Anal.*, 170(2):137–184, 2003.

[10] L. C. Evans and R. Gariepy. Measure Theory and Fine Properties of Functions. Studies in Advanced Mathematics, 1992.

[11] H. Federer. Geometric Measure Theory. Die Grundlehren der Mathematischen Wissenschaften, Band, 153, Springer-Verlag New York Inc.: New york, 1969.
[12] E. Giusti. Minimal surfaces and functions of bounded variation. *Monographs in mathematics*, 80: Boston, 1984.
[13] Y. Kwon. Strong traces for multidimensional scalar conservation laws. *Preprint.*
[14] E. Yu. Panov. Existence of strong traces for generalized solutions of multidimensional scalar conservation laws. *Journal of Hyperbolic Differential Equations*, 2(4):885–908, 2005.
[15] N. C. Phuc and M. Torres. Characterizations of the existence and removable singularities of divergence-measure vector fields. *Indiana Univ. Math. J.*, 2008 (to appear).
[16] W. Rudin. Principi di Analisi Matematica. *McGraw-Hill* , 1991.
[17] M. Silhavy. Divergence measure fields and Cauchy's stress theorem. *Rend. Sem. Mat. Padova*, 113:15–45, 2005.
[18] A. Vasseur. Strong traces for solutions of multidimensional scalar conservation laws. *Arch. Rational Mech. Anal.*, 160:181–193, 2001.
[19] A. Vasseur and Y. Kwon. Strong traces for solutions to scalar conservation laws with general flux. *Arch. Rational Mech. Anal.*, 185(3):495–513, 2007.
[20] W. Ziemer. Weakly differentiable funtions. *Graduate Texts in Mathematics*, 120, Springer-Verlag: New York, 1989.

(M. Torres) Department of Mathematics, Purdue University, 150 N. University Street 47907-2067, USA. http://www.math.purdue.edu/ torres
E-mail address: torres@math.purdue.edu