ON THE DISTRIBUTIONAL DIVERGENCE OF VECTOR FIELDS VANISHING AT INFINITY

THIERRY DE PAUW AND MONICA TORRES

Abstract. The equation $\operatorname{div} v = F$ has a solution v in the space of continuous vector fields vanishing at infinity if and only if F acts linearly on $BV_{\frac{m}{m-1}}(\mathbb{R}^m)$ (the space of functions in $L^{\frac{m}{m-1}}(\mathbb{R}^m)$ whose distributional gradient is a vector valued measure) and satisfies the following continuity condition: $F(u_j)$ converges to zero for each sequence $\{u_j\}$ such that the measure norms of ∇u_j are uniformly bounded and $u_j \rightharpoonup 0$ weakly in $L^{\frac{m}{m-1}}(\mathbb{R}^m)$.

1. Introduction

The equation $\triangle u = f \in L^m(\mathbb{R}^m)$ need not have a solution $u \in C^1(\mathbb{R}^m)$. In this paper we prove that to each $f \in L^m(\mathbb{R}^m)$ there corresponds a continuous vector field, vanishing at infinity, $v \in C_0(\mathbb{R}^m;\mathbb{R}^m)$ such that $\operatorname{div} v = f$ weakly. In fact we characterize those distributions F on $\mathbb{R}^m$ such that the equation $\operatorname{div} v = F$ admits a weak solution $v \in C_0(\mathbb{R}^m;\mathbb{R}^m)$. Related results have been obtained in [1], [3], [4], [2] and [6]. Our first proof, contained in sections 3 to 6, follows the same pattern as [3]. A second proof, presented in section 7, is based on the more abstract methods developed in [2].

In this paper $m \geqslant 2$ and $1^* := m/(m-1)$. Let $BV_{1^*}(\mathbb{R}^m)$ denote the subspace of $L^{1^*}(\mathbb{R}^m)$ consisting of those functions u whose distributional gradient ∇u is a vector valued measure (of finite total mass). We define a charge vanishing at infinity to be a linear functional $F : BV_{1^*}(\mathbb{R}^m) \to \mathbb{R}$ such that $F(u_j) \to 0$ whenever

$$u_j \rightharpoonup 0 \text{ weakly in } L^{1^*}(\mathbb{R}^m) \text{ and } \sup_j \|\nabla u\|_{\mathscr{M}} < \infty\,. \tag{1}$$

We denote by $CH_0(\mathbb{R}^m)$ the space of charges vanishing at infinity and we note (Proposition 3.2) it is a closed subspace of the dual of $BV_{1^*}(\mathbb{R}^m)$ (where the latter is equiped with its norm $\|\nabla u\|_{\mathscr{M}}$). Examples of charges vanishing at infinity include the functions $f \in L^m(\mathbb{R}^m)$ (Proposition 3.4) and the distributional divergence $\operatorname{div} v$ of $v \in C_0(\mathbb{R}^m;\mathbb{R}^m)$ (Proposition 3.5). Our main result thus consists in showing that the operator

$$C_0(\mathbb{R}^m;\mathbb{R}^m) \to CH_0(\mathbb{R}^m) : v \mapsto \operatorname{div} v \tag{2}$$

is onto. This is done by applying the Closed Range Theorem. For this purpose we identify $CH_0(\mathbb{R}^m)^*$ with $BV_{1^*}(\mathbb{R}^m)$ via the evaluation map (Proposition 5.1). This in turn relies on the fact that $L^m(\mathbb{R}^m)$ is dense in $CH_0(\mathbb{R}^m)$ (Corollary 4.3, obtained by smoothing). Therefore the adjoint of (2) is

$$BV_{1^*}(\mathbb{R}^m) \to \mathscr{M}(\mathbb{R}^m;\mathbb{R}^m) : u \mapsto -\nabla u\,.$$

That this operator have a closed range follows from compactness in $BV_{1^*}(\mathbb{R}^m)$ (Proposition 2.6).

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Charges vanishing at infinity happen to be the linear functionals on $BV_{1^*}(\mathbb{R}^m)$ which are continuous with respect to a certain locally convex linear (sequential, non metrizable, non barrelled) topology $\mathfrak{T}_{\mathcal{C}}$ on $BV_{1^*}(\mathbb{R}^m)$. In other words there exists a locally convex topology $\mathfrak{T}_{\mathcal{C}}$ on $BV_{1^*}(\mathbb{R}^m)$ such that a sequence $u_j \to 0$ in the sense of $\mathfrak{T}_{\mathcal{C}}$ if and only the sequence $\{u_j\}$ verifies the conditions of (1). Topologies of this type have been studied in [2, section 3]. Referring to the general theory yields a quicker, though very much abstract proof in Section 7. In order to appreciate this alternative route the reader is expected to be familiar with the methods of [2, section 3]. From this perspective the key identification $CH_0(\mathbb{R}^m)^* \cong BV_{1^*}(\mathbb{R}^m)$ is simply saying that $BV_{1^*}(\mathbb{R}^m)[\mathfrak{T}_{\mathcal{C}}]$ is semireflexive, a property which follows from the Compactness Proposition 2.6.

2. Preliminaries

A continuous vector field $v : \mathbb{R}^m \to \mathbb{R}^m$ is said to *vanish at infinity* if for every $\varepsilon > 0$ there exists a compact set $K \subseteq \mathbb{R}^m$ such that $|v(x)| \leqslant \varepsilon$ whenever $x \in \mathbb{R}^m \setminus K$. These form a linear space denoted $C_0(\mathbb{R}^m;\mathbb{R}^m)$ which is complete under the norm $\|v\|_\infty := \sup\{|v(x)| : x \in \mathbb{R}^m\}$. The linear subspace $C_c(\mathbb{R}^m;\mathbb{R}^m)$ (resp. $\mathscr{D}(\mathbb{R}^m;\mathbb{R}^m)$) consisting of those vector fields having compact support (resp. smooth vector fields having compact support) is dense in $C_0(\mathbb{R}^m;\mathbb{R}^m)$. Thus each element of the dual, $T \in C_0(\mathbb{R}^m;\mathbb{R}^m)^*$, is uniquely associated with some vector valued measure $\mu \in \mathscr{M}(\mathbb{R}^m;\mathbb{R}^m)$ as follows:

$$T(v) = \int_{\mathbb{R}^m} \langle v, d\mu \rangle \,,$$

according to the Riesz-Markov representation Theorem. Furthermore,

$$\|\mu\|_{\mathscr{M}} = \sup\left\{ \int_{\mathbb{R}^m} \langle v, d\mu\rangle : v \in \mathscr{D}(\mathbb{R}^m;\mathbb{R}^m) \text{ and } \|v\|_\infty \leqslant 1 \right\} .$$

A vector valued distribution $T \in \mathscr{D}(\mathbb{R}^m;\mathbb{R}^m)^*$ with the property that

$$\sup\{T(v) : v \in \mathscr{D}(\mathbb{R}^m;\mathbb{R}^m) \text{ and } \|v\|_\infty \leqslant 1\} < \infty$$

extends uniquely to an element of $C_0(\mathbb{R}^m;\mathbb{R}^m)$ and thus is associated with a vector valued measure as above.

We recall some properties of convolution. Let $1 \leqslant p < \infty$, $u \in L^p(\mathbb{R}^m)$, and $\varphi \in \mathscr{D}(\mathbb{R}^m)$. For each $x \in \mathbb{R}^m$ we define

$$(u * \varphi)(x) = \int_{\mathbb{R}^m} u(y)\varphi(x-y)dy \,.$$

It follows from Young's inequality that $u * \varphi \in L^p(\mathbb{R}^m)$ and

$$\|u * \varphi\|_{L^p} \leqslant \|u\|_{L^p} \|\varphi\|_{L^1} \,. \tag{3}$$

Furthermore $u * \varphi \in C^\infty(\mathbb{R}^m)$ and $\nabla(u * \varphi) = u * \nabla\varphi$. In case φ is even and $f \in L^q(\mathbb{R}^m)$ with $p^{-1} + q^{-1} = 1$ we have

$$\int_{\mathbb{R}^m} f(u * \varphi) = \int_{\mathbb{R}^m} u(f * \varphi) \,.$$

We fix an *approximate identity on* $\mathbb{R}^m$, $\{\varphi_k\}$ (see [5, 6.31]), and we infer that

$$\lim_k \|u - u * \varphi_k\|_{L^p} = 0 \,. \tag{4}$$

In the remaining part of this paper we assume that $m \geqslant 2$. We let the Sobolev conjugate exponent of 1 be

$$1^* := \frac{m}{m-1} \,.$$

Notice that $L^{1^*}(\mathbb{R}^m)$ is isometrically isomorphic to $L^m(\mathbb{R}^m)^*$. We recall the Gagliardo-Nirenberg-Sobolev inequality:

$$\|\varphi\|_{L^{1^*}} \leqslant \kappa_m \|\nabla\varphi\|_{L^1}$$

whenever $\varphi \in \mathscr{D}(\mathbb{R}^m)$.

Definition 2.1. — *We let $BV_{1^*}(\mathbb{R}^m)$ denote the linear subspace of $L^{1^*}(\mathbb{R}^m)$ consisting of those functions u whose distributional gradient ∇u is a vector valued measure, i.e.*

$$\|\nabla u\|_{\mathscr{M}} = \sup\left\{\int_{\mathbb{R}^m} u \operatorname{div} v : v \in \mathscr{D}(\mathbb{R}^m;\mathbb{R}^m) \text{ and } \|v\|_\infty \leqslant 1\right\} < \infty\,.$$

Readily $|||u||| := \|u\|_{L^{1^*}} + \|\nabla u\|_{\mathscr{M}}$ defines a norm on $BV_{1^*}(\mathbb{R}^m)$ which makes it into a Banach space. In view of Proposition 2.5, we will use in this paper the equivalent norm $\|u\|_{BV_{1^*}} := \|\nabla u\|_{\mathscr{M}}$

Definition 2.2. — *Given a sequence $\{u_j\}$ in $BV_{1^*}(\mathbb{R}^m)$ we write $u_j \twoheadrightarrow 0$ whenever*

1. $\sup_j \|\nabla u_j\|_{\mathscr{M}} < \infty$;
2. $u_j \rightharpoonup 0$ *weakly in* $L^{1^*}(\mathbb{R}^m)$.

Proposition 2.3. — *Let $\{u_j\}$ be a sequence in $BV_{1^*}(\mathbb{R}^m)$, $u \in L^{1^*}(\mathbb{R}^m)$, and assume that $u_j \rightharpoonup u$ weakly in $L^{1^*}(\mathbb{R}^m)$. It follows that*

$$\|\nabla u\|_{\mathscr{M}} \leqslant \liminf_j \|\nabla u_j\|_{\mathscr{M}}\,. \tag{5}$$

Proof. Let $v \in \mathscr{D}(\mathbb{R}^m;\mathbb{R}^m)$ with $\|v\|_\infty \leqslant 1$. Since $\operatorname{div} v \in L^m(\mathbb{R}^m)$ and $u_j \rightharpoonup u$ weakly in $L^{1^*}(\mathbb{R}^m)$ we have from Definition 2.1,

$$\int_{\mathbb{R}^m} u \operatorname{div} v = \lim_j \int_{\mathbb{R}^m} u_j \operatorname{div} v \leqslant \liminf_j \|\nabla u_j\|_{\mathscr{M}},$$

and taking the supremum over all such v we conclude

$$\|\nabla u\|_{\mathscr{M}} \leqslant \liminf_j \|\nabla u_j\|_{\mathscr{M}}.$$

□

The following density result is basic.

Proposition 2.4. — *Let $u \in BV_{1^*}(\mathbb{R}^m)$. The following hold.*

(A) *For every $\varphi \in \mathscr{D}(\mathbb{R}^m)$, $u * \varphi \in BV_{1^*}(\mathbb{R}^m)$ and*

$$\|\nabla(u*\varphi)\|_{L^1} \leqslant \|\nabla u\|_{\mathscr{M}} \|\varphi\|_{L^1}\,;$$

(B) *If $\{\varphi_k\}$ is an approximate identity then*

$$u - u*\varphi_k \twoheadrightarrow 0 \text{ and } \lim_k \|\nabla(u*\varphi_k)\|_{L^1} = \|\nabla u\|_{\mathscr{M}}\,;$$

(C) *There exists a sequence $\{u_j\}$ in $\mathscr{D}(\mathbb{R}^m)$ such that*

$$u - u_j \twoheadrightarrow 0 \text{ as well as } \lim_j \|\nabla u_j\|_{L^1} = \|\nabla u\|_{\mathscr{M}}\,.$$

Proof. We note that (3) yields $u*\varphi \in L^{1^*}$. We have

$$\begin{aligned}\int_{\mathbb{R}^m} |\nabla(u*\varphi)|(x)dx &= \int_{\mathbb{R}^m} |\varphi * \nabla u|(x)dx = \int_{\mathbb{R}^m}\left|\int_{\mathbb{R}^m}\varphi(x-y)d\nabla u(y)\right|dx \\ &\leqslant \int_{\mathbb{R}^m}\int_{\mathbb{R}^m}|\varphi(x-y)|d\|\nabla u\|(y)dx = \int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m}|\varphi(x-y)|dx\right)d\|\nabla u\|(y) \\ &= \|\nabla u\|_{\mathscr{M}}\|\varphi\|_{L^1},\end{aligned} \tag{6}$$

which shows (A).

Let $\{\varphi_k\}$ be an approximate identity. From (A) we obtain

$$\|\nabla(u*\varphi_k)\|_{\mathscr{M}} = \int_{\mathbb{R}^m} |\nabla(u*\varphi_k)|(x)dx \leqslant \|\nabla u\|_{\mathscr{M}}\|\varphi_k\|_{L^1} = \|\nabla u\|_{\mathscr{M}}. \tag{7}$$

Since $u*\varphi_k \to u$ in $L^{1^*}(\mathbb{R}^m)$ then in particular $u*\varphi_k \rightharpoonup u$ weakly in $L^{1^*}(\mathbb{R}^m)$; i.e.

$$\int_{\mathbb{R}^m} f[(u*\varphi_k) - u] \to 0 \text{ for every } f \in L^m(\mathbb{R}^m). \tag{8}$$

From (7) and (8) we obtain that $u - u*\varphi_k \twoheadrightarrow 0$. Moreover, from (7) and the lower semicontinuity (5) we conclude that $\lim_k \|\nabla(u*\varphi_k)\|_{L^1} = \|\nabla u\|_{\mathscr{M}}$, which shows (B) holds.

In order to establish (C), we choose a sequence $\{\psi_i\}$ in $\mathscr{D}(\mathbb{R}^m)$ such that

$$\mathbb{1}_{B(0,i)} \leqslant \psi_i \leqslant \mathbb{1}_{B(0,2i)} \quad \text{and} \quad \sup_i \|\nabla\psi_i\|_{L^m} < \infty. \tag{9}$$

As usual let $\{\varphi_k\}$ be an approximate identity. Referring to (B) we define inductively a strictly increasing sequence of integers $\{k_j\}$ such that

$$\int_{\mathbb{R}^m} |\nabla(u*\varphi_{k_j})| \leqslant \|\nabla u\|_{\mathscr{M}} + \frac{1}{j}.$$

For each j and i we observe that

$$\left|\nabla\left[(u*\varphi_{k_j})\psi_i\right]\right| \leqslant \left|\psi_i\nabla(u*\varphi_{k_j})\right| + \left|(u*\varphi_{k_j})\nabla\psi_i\right| .$$

For fixed j we infer from (9) and the relation $|u*\varphi_{k_j}|^{1^*} \in L^1(\mathbb{R}^m)$ that

$$\begin{aligned}\limsup_i \int_{\mathbb{R}^m} |(u*\varphi_{k_j})\nabla\psi_i| &= \limsup_i \int_{B(0,i)^c} |(u*\varphi_{k_j})\nabla\psi_i| \\ &\leqslant \limsup_i \left(\int_{B(0,i)^c} |u*\varphi_{k_j}|^{1^*}\right)^{1/1^*} \|\nabla\psi_i\|_{L^m} \\ &= 0\,.\end{aligned}$$

According to the three preceding inequalities we can define inductively a strictly increasing sequence of integers $\{i_j\}$ such that

$$\int_{\mathbb{R}^m} \left|\nabla\left[(u*\varphi_{k_j})\psi_{i_j}\right]\right| \leqslant \int_{\mathbb{R}^m} |\nabla(u*\varphi_{k_j})| + \frac{1}{j} \leqslant \|\nabla u\|_{\mathscr{M}} + \frac{2}{j}\,.$$

We put $u_j := (u*\varphi_{k_j})\psi_{i_j}$. In view of Proposition 2.3 it remains only to show that $u_j \rightharpoonup u$ weakly in $L^{1^*}(\mathbb{R}^m)$. Given $f \in L^m(\mathbb{R}^m)$ we notice that

$$\begin{aligned}\left|\int_{\mathbb{R}^m} f\left(u - (u*\varphi_{k_j})\psi_{i_j}\right)\right| &\leqslant \int_{\mathbb{R}^m} |f||u-(u*\varphi_{k_j})| + \int_{\mathbb{R}^m} |f||u*\varphi_{k_j}||1-\psi_{i_j}| \\ &\leqslant \|f\|_{L^m}\|u-(u*\varphi_{k_j})\|_{L^{1^*}} \\ &\quad + \left(\int_{B(0,i_j)^c} |f|^m\right)^{1/m} \|u\|_{L^{1^*}}\|\varphi_{k_j}\|_{L^1}\,.\end{aligned}$$

The latter tends to zero as $j \to \infty$ and the proof is complete. ☐

Proposition 2.5 (Gagliardo-Nirenberg-Sobolev inequality). — *Let $u \in BV_{1^*}(\mathbb{R}^m)$. One has*

$$\|u\|_{L^{1^*}} \leqslant \kappa_m \|\nabla u\|_{\mathscr{M}}\,.$$

Proof. Since the norm $\|\cdot\|_{L^{1^*}}$ in $L^{1^*}(\mathbb{R}^m)$ is lower semicontinuous with respect to weak convergence the result is a consequence of Proposition 2.4(C) and the Gagliardo-Nirenberg-Sobolev inequality for functions in $\mathscr{D}(\mathbb{R}^m)$. ☐

PROPOSITION 2.6 (Compactness). — *Let $\{u_j\}$ be a bounded sequence in $BV_{1^*}(\mathbb{R}^m)$, i.e. $\sup_j \|\nabla u_j\|_{\mathscr{M}} < \infty$. There then exist a subsequence $\{u_{j_k}\}$ of $\{u_j\}$ and $u \in BV_{1^*}(\mathbb{R}^m)$ such that $u_{j_k} - u \twoheadrightarrow 0$.*

Proof. Since $\{u_j\}$ is bounded in $BV_{1^*}(\mathbb{R}^m)$ it is also bounded in $L^{1^*}(\mathbb{R}^m)$ according to Proposition 2.5. The conclusion thus immediately follows from the fact that $L^{1^*}(\mathbb{R}^m)$ is a reflexive Banach space whose dual is separable, together with Proposition 2.3. □

3. Charges vanishing at infinity

DEFINITION 3.1. — *A* charge vanishing at infinity *is a linear functional $F : BV_{1^*}(\mathbb{R}^m) \to \mathbb{R}$ such that $\langle u_j, F\rangle \to 0$ whenever $u_j \twoheadrightarrow 0$. The collection of these is denoted $CH_0(\mathbb{R}^m)$.*

One readily sees that $CH_0(\mathbb{R}^m)$ is a linear space. With $F \in CH_0(\mathbb{R}^m)$ we associate

$$\|F\|_{CH_0} := \sup\{\langle u, F\rangle : u \in BV_{1^*}(\mathbb{R}^m) \text{ and } \|\nabla u\|_{\mathscr{M}} \leqslant 1\}\,.$$

One checks that $\|F\|_{CH_0} < \infty$ for each $F \in CH_0(\mathbb{R}^m)$ according to Proposition 2.6, hence $\|\cdot\|_{CH_0}$ is a norm on $CH_0(\mathbb{R}^m)$. Notice that $CH_0(\mathbb{R}^m) \subseteq BV_{1^*}(\mathbb{R}^m)^*$ and $\|F\|_{CH_0} = \|F\|_{(BV_{1^*})^*}$ whenever $F \in CH_0(\mathbb{R}^m)$.

PROPOSITION 3.2. — *$CH_0(\mathbb{R}^m)[\|\cdot\|_{CH_0}]$ is a Banach space.*

Proof. Let $\{F_k\}$ be a Cauchy sequence in $CH_0(\mathbb{R}^m)$. It follows that $\{F_k\}$ converges in $BV_{1^*}(\mathbb{R}^m)^*$ to some $F \in BV_{1^*}(\mathbb{R}^m)^*$ and it remains only to check that F is a charge vanishing at infinity. Let $\{u_j\}$ be a sequence in $BV_{1^*}(\mathbb{R}^m)$ such that $u_j \twoheadrightarrow 0$ and put $\Gamma := \sup_j \|\nabla u_j\|_{\mathscr{M}}$. Given $\varepsilon > 0$ choose an integer k such that $\|F - F_k\|_{BV_{1^*}^*} \leqslant \varepsilon$. Observe that for each j

$$\begin{aligned}|\langle u_j, F\rangle| &\leqslant |\langle u_j, F_k\rangle| + |\langle u_j, F - F_k\rangle| \\ &\leqslant |\langle u_j, F_k\rangle| + \|F - F_k\|_{BV_{1^*}^*}\Gamma \\ &\leqslant |\langle u_j, F_k\rangle| + \varepsilon\Gamma\,.\end{aligned}$$

Thus $\limsup_j |\langle u_j, F\rangle| \leqslant \varepsilon\Gamma$ and since ε is arbitrary the conclusion follows. □

The following is a justification for the terminology "vanishing at infinity".

PROPOSITION 3.3. — *Let $F \in CH_0(\mathbb{R}^m)$ and $\varepsilon > 0$. There then exists a compact set $K \subseteq \mathbb{R}^m$ such that $|\langle u, F\rangle| \leqslant \varepsilon\|\nabla u\|_{\mathscr{M}}$ whenever $u \in BV_{1^*}(\mathbb{R}^m)$ and $K \cap \operatorname{supp} u = \emptyset$.*

Proof. Let $F \in CH_0(\mathbb{R}^m)$. Assume if possible that there exist $\varepsilon > 0$ and a sequence $\{u_j\}$ in $BV_{1^*}(\mathbb{R}^m)$ such that $\|\nabla u_j\|_{\mathscr{M}} = 1$, $B(0,j) \cap \operatorname{supp} u_j = \emptyset$, and $|\langle u_j, F\rangle| \geqslant \varepsilon$ for every j. We claim that $u_j \twoheadrightarrow 0$. In order to show this it suffices to establish that $u_j \rightharpoonup 0$ weakly in $L^{1^*}(\mathbb{R}^m)$. Let $f \in L^m(\mathbb{R}^m)$. Given $\eta > 0$ there exists a compact set $K \subseteq \mathbb{R}^m$ such that $\int_{\mathbb{R}^m \setminus K} |f|^m \leqslant \eta^m$. If j is sufficiently large for $K \subseteq B(0,j)$ then

$$\left|\int_{\mathbb{R}^m} f u_j\right| = \left|\int_{\mathbb{R}^m \setminus K} f u_j\right| \leqslant \left(\int_{\mathbb{R}^m \setminus K} |f|^m\right)^{1/m} \|u_j\|_{L^{1^*}} \leqslant \eta\kappa_m\,.$$

Thus $\limsup_j \left|\int_{\mathbb{R}^m} f u_j\right| \leqslant \eta\kappa_m$ and since η is arbitrary we infer that $\int_{\mathbb{R}^m} f u_j \to 0$. This establishes our claim and in turn implies that $\lim_j \langle u_j, F\rangle = 0$, a contradiction. □

We now turn to giving the two main examples of charges vanishing at infinity. Given $f \in L^m(\mathbb{R}^m)$ we define (recall that $BV_{1^*}(\mathbb{R}^m) \subseteq L^{1^*}(\mathbb{R}^m)$)

$$\Lambda(f) : BV_{1^*}(\mathbb{R}^m) \to \mathbb{R} : u \mapsto \int_{\mathbb{R}^m} uf\,.$$

Proposition 3.4. — *Given $f \in L^m(\mathbb{R}^m)$ one has $\Lambda(f) \in CH_0(\mathbb{R}^m)$ and*

$$\|\Lambda(f)\|_{CH_0} \leqslant \kappa_m \|f\|_{L^m}\,.$$

Thus

$$\Lambda : L^m(\mathbb{R}^m) \to CH_0(\mathbb{R}^m)$$

is a bounded linear operator.

Proof. Let $\{u_j\}$ be a sequence in $BV_{1^*}(\mathbb{R}^m)$ such that $u_j \twoheadrightarrow 0$. Then $u_j \rightharpoonup 0$ weakly in $L^{1^*}(\mathbb{R}^m)$ whence $\langle u_j, \Lambda(f)\rangle \to 0$, thereby showing that $\Lambda(f) \in CH_0(\mathbb{R}^m)$. Given $u \in BV_{1^*}(\mathbb{R}^m)$ we notice that

$$|\langle u, \Lambda(f)\rangle| \leqslant \|u\|_{L^{1^*}} \|f\|_{L^m} \leqslant \kappa_m \|\nabla u\|_{\mathscr{M}} \|f\|_{L^m}$$

so that $\|\Lambda(f)\|_{CH_0} \leqslant \kappa_m \|f\|_{L^m}$. □

Given $v \in C_0(\mathbb{R}^m;\mathbb{R}^m)$ and $u \in BV_{1^*}(\mathbb{R}^m)$ we notice that v is summable with respect to the measure ∇u. Thus we may define

$$\Phi(v) : BV_{1^*}(\mathbb{R}^m) \to \mathbb{R} : u \mapsto -\int_{\mathbb{R}^m} \langle v, d(\nabla u)\rangle\,.$$

Proposition 3.5. — *Given $v \in C_0(\mathbb{R}^m;\mathbb{R}^m)$ one has $\Phi(v) \in CH_0(\mathbb{R}^m)$ and*

$$\|\Phi(v)\|_{CH_0} \leqslant \|v\|_\infty\,.$$

Thus

$$\Phi : C_0(\mathbb{R}^m;\mathbb{R}^m) \to CH_0(\mathbb{R}^m)$$

is a bounded linear operator.

Proof. Let $v \in C_0(\mathbb{R}^m;\mathbb{R}^m)$ and let $\{u_j\}$ be a sequence in $BV_{1^*}(\mathbb{R}^m)$ such that $u_j \twoheadrightarrow 0$. Given $\varepsilon > 0$ we choose $w \in \mathscr{D}(\mathbb{R}^m;\mathbb{R}^m)$ such that $\|w - v\|_\infty \leqslant \varepsilon$. Put $\Gamma = \sup_j \|\nabla u_j\|_{\mathscr{M}}$. We notice that

$$|\langle u_j, \Phi(v)\rangle| \leqslant \left| \int_{\mathbb{R}^m} \langle v - w, d(\nabla u_j)\rangle \right| + \left| \int_{\mathbb{R}^m} u_j \operatorname{div} w \right| \leqslant \varepsilon\Gamma + \left| \int_{\mathbb{R}^m} u_j \operatorname{div} w \right|\,.$$

Since $\operatorname{supp} \operatorname{div} w$ is compact we infer that $\operatorname{div} w \in L^m(\mathbb{R}^m)$, hence $\lim_j \int_{\mathbb{R}^m} u_j \operatorname{div} w = 0$. Thus $\limsup_j |\langle u_j, \Phi(v)\rangle| \leqslant \varepsilon\Gamma$ and from the arbitrariness of ε we conclude that $\Phi(v) \in CH_0(\mathbb{R}^m)$.

Finally if $u \in BV_{1^*}(\mathbb{R}^m)$ then

$$|\langle u, \Phi(v)\rangle| = \left| \int_{\mathbb{R}^m} \langle v, d(\nabla u)\rangle \right| \leqslant \|v\|_\infty \|\nabla u\|_{\mathscr{M}}\,,$$

thus $\|\Phi(v)\|_{CH_0} \leqslant \|v\|_\infty$. □

4. Approximation

Let $F \in CH_0(\mathbb{R}^m)$ and $\varphi \in \mathscr{D}(\mathbb{R}^m)$. Our goal is to define a new charge vanishing at infinity, the convolution of F and φ, denoted $F * \varphi$, to show it belongs to the range of Λ (Proposition 3.4), and that it approximates F in the norm $\|\cdot\|_{CH_0}$. We start by observing that if $u \in BV_{1^*}(\mathbb{R}^m)$ then $u * \varphi \in BV_{1^*}(\mathbb{R}^m)$ (Proposition 2.4(A)). Therefore the following is a well-defined linear functional:

$$F * \varphi : BV_{1^*}(\mathbb{R}^m) \to \mathbb{R} : u \mapsto \langle u * \varphi, F\rangle\,.$$

We now show that $F * \varphi$ is indeed a charge vanishing at infinity, in fact of the special type $\Lambda(f)$ for some $f \in L^m(\mathbb{R}^m)$. We denote by $\mathscr{R}(\Lambda)$ the range of the operator Λ.

Proposition 4.1. — *Let $F \in CH_0(\mathbb{R}^m)$ and $\varphi \in \mathscr{D}(\mathbb{R}^m)$. It follows that $F * \varphi \in CH_0(\mathbb{R}^m) \cap \mathscr{R}(\Lambda)$.*

Proof. The restriction of F to $\mathscr{D}(\mathbb{R}^m)$ is a distribution, still denoted F, thus the convolution $F * \varphi$ is associated with a smooth function $f \in C^\infty(\mathbb{R}^m)$ as follows:

$$\langle \psi, F * \varphi \rangle = \int_{\mathbb{R}^m} \psi f \tag{10}$$

for every $\psi \in \mathscr{D}(\mathbb{R}^m)$ (see e.g. [5, 6.30(b)]). We claim that $f \in L^m(\mathbb{R}^m)$. Let $\{\psi_j\}$ be a sequence in $\mathscr{D}(\mathbb{R}^m)$ such that $\|\psi_j\|_{L^{1^*}} \to 0$. Notice that

$$\sup_j \|\nabla(\psi_j * \varphi)\|_{\mathscr{M}} = \sup_j \|\nabla(\psi_j * \varphi)\|_{L^1} \leqslant \sup_j \|\psi_j\|_{L^{1^*}} \|\nabla\varphi\|_{L^q} < \infty\,,$$

where $q = m/(m+1)$, according to Young's inequality. For any $g \in L^m(\mathbb{R}^m)$ we have $\int_{\mathbb{R}^m} g(\psi_j * \varphi) = \int_{\mathbb{R}^m} \psi_j(g * \varphi) \to 0$ since $g * \varphi \in L^m(\mathbb{R}^m)$ and $\psi_j \rightharpoonup 0$ weakly in L^{1^*}. Therefore, $\psi_j * \varphi \twoheadrightarrow 0$ and in turn $\langle \psi_j * \varphi, F\rangle = \langle \psi_j, F * \varphi\rangle \to 0$. This shows that $F * \varphi$ is $\|\cdot\|_{L^{1^*}}$ continuous in $\mathscr{D}(\mathbb{R}^m)$. Since $\mathscr{D}(\mathbb{R}^m)$ is dense in $L^{1^*}(\mathbb{R}^m)$ we infer that $F * \varphi$ can be uniquely extended to a continuous linear functional on $L^{1^*}(\mathbb{R}^m)$. Thus, the Riesz representation theorem yields $f \in L^m(\mathbb{R}^m)$ and therefore Proposition 3.4 gives $\Lambda(f) \in CH_0(\mathbb{R}^m)$. It only remains to show that $\Lambda(f) = F * \varphi$, which is equivalent to showing that equation (10) actually holds for every $\psi \in BV_{1^*}(\mathbb{R}^m)$. To see this we use Proposition 2.4 (C) to obtain a sequence $\{\psi_j\} \in \mathscr{D}(\mathbb{R}^m)$ such that $\psi_j \twoheadrightarrow \psi$. Note that equation (10) holds for each ψ_j and the result follows by noting that $\int_{\mathbb{R}^m} \psi_j f \to \int_{\mathbb{R}^m} \psi f$ and $\langle \psi_j, F * \varphi\rangle = \langle \psi_j * \varphi, F\rangle \to \langle \psi * \varphi, F\rangle = \langle \psi, F * \varphi\rangle$, since $\psi_j \rightharpoonup \psi$ weakly in $L^{1^*}(\mathbb{R}^m)$ and $\psi_j * \varphi \twoheadrightarrow \psi * \varphi$.

□

It remains to show that $F * \varphi$ is a good approximation of F in $CH_0(\mathbb{R}^m)$ provided φ is a good approximation of the identity.

Proposition 4.2. — *Let $F \in CH_0(\mathbb{R}^m)$ and let $\{\varphi_k\}$ be an approximate identity such that each φ_k is even. It follows that*

$$\lim_k \|F - F * \varphi_k\|_{CH_0(\mathbb{R}^m)} = 0\,.$$

Proof. In order to simplify the notations we put $F_k = F * \varphi_k$. Since $F \in CH_0(\mathbb{R}^m)$ the following holds. For every $\varepsilon > 0$ there are $f_1, \ldots, f_J \in L^m(\mathbb{R}^m)$ and positive real numbers $\eta_1, \ldots, \eta_J$ such that $|\langle u, F\rangle| \leqslant \varepsilon$ whenever $u \in BV_{1^*}(\mathbb{R}^m)$, $\|\nabla u\|_{\mathscr{M}} \leqslant 2$ and

$$\left| \int_{\mathbb{R}^m} u f_j \right| \leqslant \eta_j$$

for every $j = 1, \ldots, J$. With each $j = 1, \ldots, J$ we associate an integer k_j such that

$$\|f_j - f_j * \varphi_k\|_{L^m} \leqslant \frac{\eta_j}{\kappa_m}$$

whenever $k \geqslant k_j$. Now given $u \in BV_{1^*}(\mathbb{R}^m)$ with $\|\nabla u\|_{\mathscr{M}} \leqslant 1$, and $k \geqslant \max\{k_1, \dots, k_J\}$, we infer that $\|\nabla(u - u * \varphi_k)\|_{\mathscr{M}} \leqslant 2$ and, for each $j = 1, \dots, J$,

$$\begin{aligned}\left|\int_{\mathbb{R}^m} (u - u * \varphi_k) f_j\right| &= \left|\int_{\mathbb{R}^m} u f_j - \int_{\mathbb{R}^m} (u * \varphi_k) f_j\right| \\ &= \left|\int_{\mathbb{R}^m} u f_j - \int_{\mathbb{R}^m} u (f_j * \varphi_k)\right| \\ &= \left|\int_{\mathbb{R}^m} u (f_j - f_j * \varphi_k)\right| \\ &\leqslant \|u\|_{L^{1^*}} \|f_j - f_j * \varphi_k\|_{L^m} \\ &\leqslant \eta_j \,.\end{aligned}$$

Therefore,

$$|\langle u, F - F_k\rangle| = |\langle u - u * \varphi_k, F\rangle| \leqslant \varepsilon \,.$$

Taking the supremum over all such u we obtain

$$\|F - F_k\| \leqslant \varepsilon$$

whenever $k \geqslant \max\{k_1, \dots, k_J\}$, and the proof is complete. □

Corollary 4.3. — *$\mathscr{R}(\Lambda)$ is dense in $CH_0(\mathbb{R}^m)$.*

5. Duality

Proposition 5.1. — *The evaluation map*

$$\mathrm{ev} : BV_{1^*}(\mathbb{R}^m) \to CH_0(\mathbb{R}^m)^*$$

is a bijection.

Proof. Since

$$\langle F, \mathrm{ev}(u)\rangle = \langle u, F\rangle$$

one readily infers that ev is injective. We now turn to proving ev is surjective. Let $\alpha \in CH_0(\mathbb{R}^m)^*$. It follows from Proposition 3.4 that $\alpha \circ \Lambda \in (L^m(\mathbb{R}^m))^*$. Thus there exists $u \in L^{1^*}(\mathbb{R}^m)$ such that

$$\langle \Lambda(f), \alpha\rangle = \int_{\mathbb{R}^m} u f \tag{11}$$

for every $f \in L^m(\mathbb{R}^m)$. Given $v \in \mathscr{D}(\mathbb{R}^m; \mathbb{R}^m)$ we notice that the charges $\Phi(v)$ (see Proposition 3.5) and $\Lambda(\operatorname{div} v)$ (see Proposition 3.4) coincide, according to Proposition 2.4(C), because they trivially coincide on $\mathscr{D}(\mathbb{R}^m)$. Thus

$$\begin{aligned}\int_{\mathbb{R}^m} u \operatorname{div} v &= \langle \Lambda(\operatorname{div} v), \alpha\rangle \\ &= \langle \Phi(v), \alpha\rangle \\ &\leqslant \|\alpha\|_{CH_0^*} \|\Phi(v)\|_{CH_0} \\ &\leqslant \|\alpha\|_{CH_0^*} \|v\|_\infty\end{aligned}$$

according to Proposition 3.5. This proves that $u \in BV_{1^*}(\mathbb{R}^m)$. It then follows from (11) that

$$\langle \Lambda(f), \alpha\rangle = \langle \Lambda(f), \mathrm{ev}(u)\rangle$$

for every $f \in L^m(\mathbb{R}^m)$. Since $\mathscr{R}(\Lambda)$ is dense in $CH_0(\mathbb{R}^m)$ (by Corollary 4.3) we conclude that $\alpha = \mathrm{ev}(u)$. □

Remark 5.2. — Notice that the evaluation map is in fact an isomorphism of the Banach spaces $BV_{1^*}(\mathbb{R}^m)[\|\cdot\|_{BV_{1^*}}]$ and $CH_0(\mathbb{R}^m)^*$, according to the Open Mapping Theorem.

6. Proof of the main theorem

Theorem 6.1. — *Let F be a distribution in $\mathbb{R}^m$. The following conditions are equivalent.*

(A) *There exists $v \in C_0(\mathbb{R}^m;\mathbb{R}^m)$ such that $\Phi(v) = F$;*
(B) *F is a charge vanishing at infinity.*

Proof. That (A) implies (B) is the content of Proposition 3.5. In order to prove that (B) implies (A) we shall first show that $\mathscr{R}(\Phi)$ is dense in $CH_0(\mathbb{R}^m)$, and then we will establish that $\mathscr{R}(\Phi)$ is closed in $CH_0(\mathbb{R}^m)$ as an application of the Closed Range Theorem.

In order to show that $\mathscr{R}(\Phi)$ is dense in $CH_0(\mathbb{R}^m)$ it suffices to prove the following, according to the Hahn-Banach Theorem: Every $\alpha \in CH_0(\mathbb{R}^m)^*$ whose restriction to $\mathscr{R}(\Phi)$ is zero, vanishes identically. Assume $\alpha \in CH_0(\mathbb{R}^m)$ and $\langle \Phi(v), \alpha\rangle = 0$ for every $v \in C_0(\mathbb{R}^m;\mathbb{R}^m)$. It ensues from Proposition 5.1 that $\alpha = \mathrm{ev}(u)$ for some $u \in BV_{1^*}(\mathbb{R}^m)$. Since $0 = \langle \Phi(v), \mathrm{ev}(u)\rangle = \int_{\mathbb{R}^m} \langle v, d(\nabla u)\rangle$ for every v we infer that $\nabla u = 0$, and in turn $u = 0$. Thus $\alpha = \mathrm{ev}(u) = 0$ and the proof that $\mathscr{R}(\Phi)$ is dense in $CH_0(\mathbb{R}^m)$ is complete.

In order to show that $\mathscr{R}(\Phi)$ is closed in $CH_0(\mathbb{R}^m)$ it suffices to show that $\mathscr{R}(\Phi^*)$ is closed in $C_0(\mathbb{R}^m;\mathbb{R}^m)^*$, according to the Closed Range Theorem. We first need to identify the adjoint map Φ^* of Φ. Recall that $CH_0(\mathbb{R}^m)^*$ is identified with $BV_{1^*}(\mathbb{R}^m)$ through the evaluation map (Proposition 5.1), and that $C_0(\mathbb{R}^m;\mathbb{R}^m)^*$ is identified with $\mathscr{M}(\mathbb{R}^m;\mathbb{R}^m)$. Given $\alpha \in CH_0(\mathbb{R}^m)^*$ we find $u \in BV_{1^*}(\mathbb{R}^m)$ such that $\alpha = \mathrm{ev}(u)$. For each $v \in C_0(\mathbb{R}^m;\mathbb{R}^m)$ one has

$$\langle v, \Phi^*(\mathrm{ev}(u))\rangle = \langle \Phi(v), \mathrm{ev}(u)\rangle = \langle u, \Phi(v)\rangle = -\int_{\mathbb{R}^m} \langle v, d(\nabla u)\rangle \,.$$

Thus $\Phi^* \circ \mathrm{ev} = -\nabla$. Now let $\{\alpha_j\}$ be a sequence in $CH_0(\mathbb{R}^m)^*$ such that $\{\Phi^*(\alpha_j)\}$ converges to some $\mu \in \mathscr{M}(\mathbb{R}^m;\mathbb{R}^m)$. We ought to prove the existence of $u \in BV_{1^*}(\mathbb{R}^m)$ such that $\mu = \nabla u$. Find a sequence $\{u_j\}$ in $BV_{1^*}(\mathbb{R}^m)$ such that $\alpha_j = \mathrm{ev}(u_j)$. Observe that

$$\|\Phi^*(\alpha_j)\|_{\mathscr{M}} = \|(\Phi^* \circ \mathrm{ev})(u_j)\|_{\mathscr{M}} = \|\nabla u_j\|_{\mathscr{M}} \,.$$

Since $\{\Phi^*(\alpha_j)\}$ is bounded we infer that $\sup_j \|\nabla u_j\|_{\mathscr{M}} < \infty$. Then, there exists a subsequence $\{u_{j_k}\}$ and $u \in BV_{1^*}(\mathbb{R}^m)$ such that $u - u_{j_k} \twoheadrightarrow 0$ according to Proposition 2.6. In particular, for each $v \in \mathscr{D}(\mathbb{R}^m;\mathbb{R}^m)$ we have

$$\int_{\mathbb{R}^m} \langle v, d(\nabla u)\rangle = -\int_{\mathbb{R}^m} u \operatorname{div} v = -\lim_k \int_{\mathbb{R}^m} u_{j_k} \operatorname{div} v = \lim_k \int_{\mathbb{R}^m} \langle v, d(\nabla u_{j_k})\rangle \,.$$

From this we infer that $\int_{\mathbb{R}^m} \langle v, d(\nabla u)\rangle = \int_{\mathbb{R}^m} \langle v, d\mu\rangle$ because μ is the limit of $\{\nabla u_j\}$. Since $\mathscr{D}(\mathbb{R}^m;\mathbb{R}^m)$ is dense in $C_0(\mathbb{R}^m;\mathbb{R}^m)$ we conclude that $\nabla u = \mu$. □

Corollary 6.2. — *For every $f \in L^m(\mathbb{R}^m)$ there exists $v \in C_0(\mathbb{R}^m;\mathbb{R}^m)$ such that $\Lambda(f) = \Phi(v)$.*

7. Another proof

Here we provide an alternative approach based on the general theory developed in [2, section 3]. Our space $X = BV_{1^*}(\mathbb{R}^m)$ is initially equiped with the locally convex linear topology $\mathfrak{T}$ which is the trace on $BV_{1^*}(\mathbb{R}^m)$ of the weak topology of $L^{1^*}(\mathbb{R}^m)$. We further consider the linearly stable family $\mathcal{C}$ (see [2, Definition 3.1]) consisting of those convex sets

$$C_j := BV_{1^*}(\mathbb{R}^m) \cap \{u : \|\nabla u\|_{\mathscr{M}} \leqslant j\} \,,$$

$j = 1, 2, \ldots$. The corresponding locally convex topology $\mathfrak{T}_{\mathcal{C}}$ on $BV_{1^*}(\mathbb{R}^m)$ is described in [2, Theorem 3.3].

We notice that the bounded subsets of $L^{1^*}(\mathbb{R}^m)$ are weakly relatively compact (according to the Banach-Alaoglu Theorem [5, 3.15], because $L^{1^*}(\mathbb{R}^m)$ is reflexive) and that the restriction of the weak topology to such subsets is metrizable (because the dual of $L^{1^*}(\mathbb{R}^m)$ is separable, see [5, 3.8(c)]). We infer from Proposition 2.5 that the sets C_j defined above are weakly bounded. Thus the C_j are $\mathfrak{T}$ relatively compact, the restriction of $\mathfrak{T}$ to C_j is sequential (in fact, metrizable) and in turn the C_j are $\mathfrak{T}$ compact according to Proposition 2.3.

We next infer from [2, Proposition 3.8(1)] that a sequence $\{u_k\}$ in $BV_{1^*}(\mathbb{R}^m)$ converges to 0 in the sense of $\mathfrak{T}_{\mathcal{C}}$ if and only if it converges to 0 in the sense of Definition 2.2. Since the restriction of $\mathfrak{T}$ to each C_j is sequential, as noted above, the proof of [2, Proposition 3.8(3)] shows that $CH_0(\mathbb{R}^m) = BV_{1^*}(\mathbb{R}^m)[\mathfrak{T}_{\mathcal{C}}]^*$.

The strong topology on $CH_0(\mathbb{R}^m)$, i.e. the topology of uniform convergence on bounded subsets of $BV_{1^*}(\mathbb{R}^m)[\mathfrak{T}_{\mathcal{C}}]$, is exactly the normed topology considered in Proposition 3.2 according to [2, Proposition 3.8(2)]. The $\mathfrak{T}$ compactness of the C_j then implies that $CH_0(\mathbb{R}^m)^* \cong BV_{1^*}(\mathbb{R}^m)$, via the evaluation map, according to [2, Theorem 3.16]. In other words Proposition 5.1 of the present paper is established in this abstract fashion. The proof of Theorem 6.1 remains unchanged.

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E-mail address: thierry.depauw@uclouvain.be

E-mail address: torres@math.purdue.edu