Section 2.6

Gradients and directional derivatives

We consider a C^1 function $f(x,y,z): \mathbb{R}^3 \to \mathbb{R}$. We will study two different important properties of ∇f.

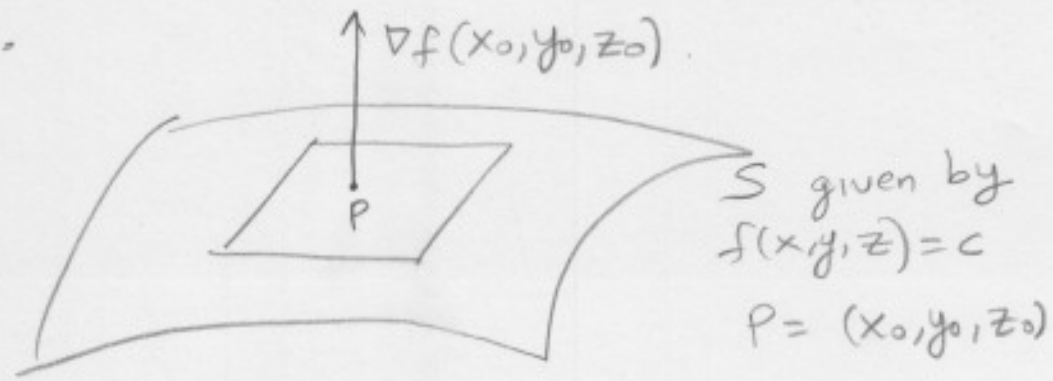
I: The first property concerns the surface determined by the equation:

$$f(x,y,z) = c,$$

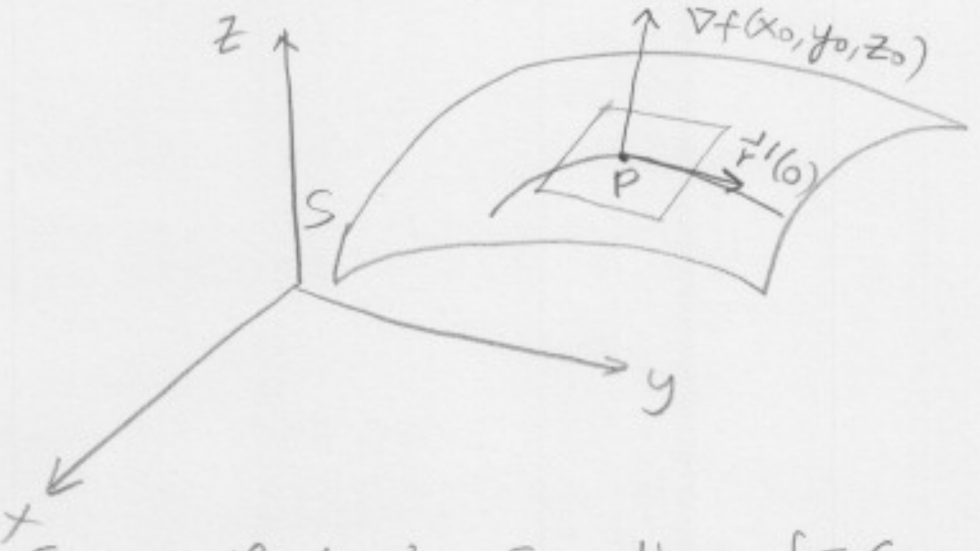
Recall that these are the level surfaces of f.

We have the following:

<u>Theorem</u>: Let S be the level surface $f(x,y,z) = c$, for some constant c. Let $(x_0,y_0,z_0) \in S$, then $\nabla f(x_0,y_0,z_0)$ is perperdicular to S, which means that $\nabla f(x_0,y_0,z_0)$ is perpendicular to the tangent plane to S at (x_0,y_0,z_0).



In order to see this, we fix a point $(x_0, y_0, z_0) \in S$ and we let $\mathcal{C}$ be a curve in S which passes through (x_0, y_0, z_0). Let $\vec{r}(t) = (x(t), y(t), z(t))$ be a parametrization for $\mathcal{C}$.



$P = (x_0, y_0, z_0)$
$\vec{r}(0) = (x_0, y_0, z_0)$

Since $\mathcal{C}$ is in S, then $f \equiv c$ on $\mathcal{C}$; ie:

$$f(x(t), y(t), z(t)) = f(\vec{r}(t)) = c$$

We differentiate with respect to t in both sides of the equation:

$$\frac{d}{dt} f(x(t), y(t), z(t)) = 0$$

Using the chain rule:

$$\frac{\partial f}{\partial x}(\vec{r}(t))\frac{dx}{dt} + \frac{\partial f}{\partial y}(\vec{r}(t))\frac{dy}{dt} + \frac{\partial f}{\partial z}(\vec{r}(t))\frac{dz}{dt} = 0$$

or

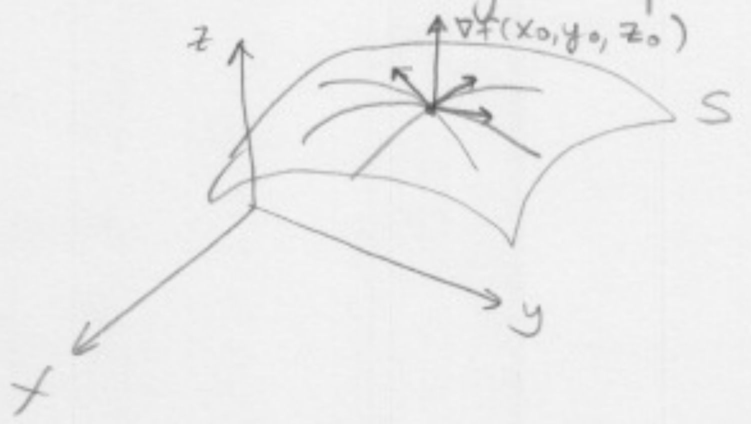
$$\nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t) = 0$$

We plug $t=0$ to get

$$\nabla f(\vec{r}(0)) \cdot \vec{r}\,'(0) = 0 \Rightarrow \nabla f(x_0, y_0, z_0) \cdot \vec{r}\,'(0) = 0$$

Since $\nabla f(x_0,y_0,z_0)\cdot\vec{r}\,'(0)=0$, and $\vec{r}\,'(0)$ is the velocity vector to $\mathcal{C}$ at (x_0,y_0,z_0), the vector $\nabla f(x_0,y_0,z_0)$ is perpendicular to $\vec{r}\,'(0)$.

Since this is true for every curve $\mathcal{C}$ through (x_0,y_0,z_0) we conclude that $\nabla f(x_0,y_0,z_0)$ is perpendicular to the tangent plane:



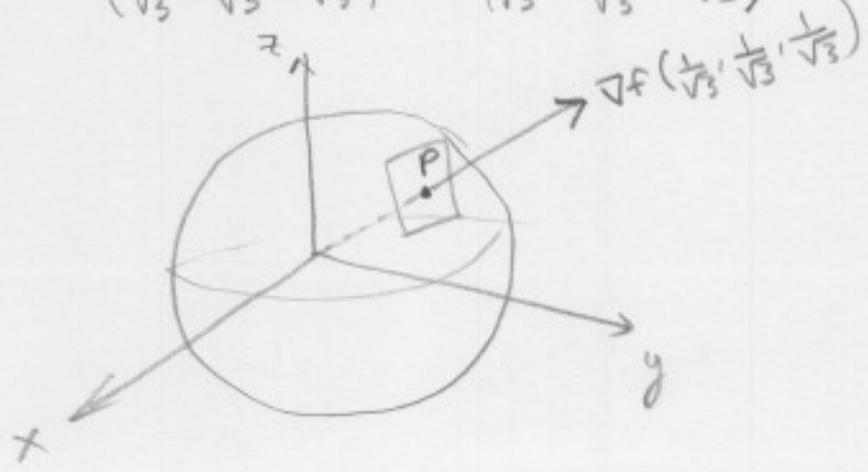
Ex: Let $f(x,y,z)=x^2+y^2+z^2$

Let S be the level surface $x^2+y^2+z^2=1$.

$\nabla f=(2x,2y,2z)$

$P=\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$ belongs to S

$\Rightarrow \nabla f\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)=\left(\frac{2}{\sqrt{3}},\frac{2}{\sqrt{3}},\frac{2}{\sqrt{3}}\right)$



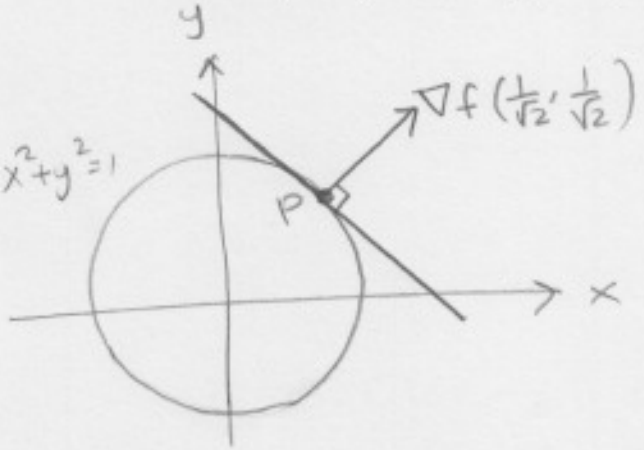
Note: The same arguments hold for $f(x,y): \mathbb{R}^2 \to \mathbb{R}$: The vector $\nabla f(x_0,y_0)$ is perpendicular to the level curve $f(x,y)=c$ that contains (x_0,y_0)

Ex: $f(x,y) = x^2+y^2$

The level curves $f(x,y)=c$ are circles.
Let $P = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, P belongs to the level curve:

$$x^2+y^2=1$$

$$\nabla f = (2x, 2y), \quad \nabla f\left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right) = \left(\tfrac{2}{\sqrt{2}}, \tfrac{2}{\sqrt{2}}\right)$$



$\nabla f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is perpendicular to the tangent line to $x^2+y^2=1$ at P. Thus, we say that $\nabla f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is perpendicalar to the level curve $x^2+y^2=1$.

Note: Since $\nabla f(x_0, y_0, z_0)$ is a normal vector to the tangent plane at (x_0, y_0, z_0), the equation of the tangent plane is:

$$\frac{\partial f}{\partial x}(x_0, y_0, z_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0, z_0)(y - y_0) + \frac{\partial f}{\partial z}(x_0, y_0, z_0)(z - z_0) = 0$$

Ex: Compute the equation of the tangent plane to $z = (\cos x)(\cos y)$ at $(0, \frac{\pi}{2}, 0)$.

We first note that $0 = (\cos 0)(\cos \frac{\pi}{2})$ ✓.

We can consider $f(x,y) = (\cos x)(\cos y)$ and compute the tangent plane as:

$$z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0).$$

Instead, we think of the graph $z = (\cos x)(\cos y)$ as the level surface at 0 of the function

$$F(x, y, z) = z - \cos x \cos y.$$

The surface S is given by $F(x, y, z) = 0$, or $z - (\cos x)(\cos y) = 0$.

$$\frac{\partial F}{\partial x} = \sin x \cos y \qquad \frac{\partial F}{\partial y} = \cos x \sin y \qquad \frac{\partial F}{\partial z} = 1$$

$$\frac{\partial F}{\partial x}(0, \tfrac{\pi}{2}, 0) = 0, \qquad \frac{\partial F}{\partial y}(0, \tfrac{\pi}{2}, 0) = 1$$

$$\therefore \quad 0\,(x - 0) + 1 \cdot (y - \tfrac{\pi}{2}) + 1 \cdot (z - 0) = 0$$

$$\Rightarrow \quad \boxed{y + z = \frac{\pi}{2}}$$

II. The second application of ∇f involves the directional derivative.

Let $\vec{u} = (u_1, u_2)$ be a <u>unit vector</u>. The directional derivative of $f(x,y)$ in the direction of $\vec{u}$ at (x_0, y_0) is:

$$\frac{\partial f}{\partial \vec{u}}(x_0,y_0) = \lim_{h\to 0} \frac{f((x_0,y_0)+h(u_1,u_2)) - f(x_0,y_0)}{h}$$

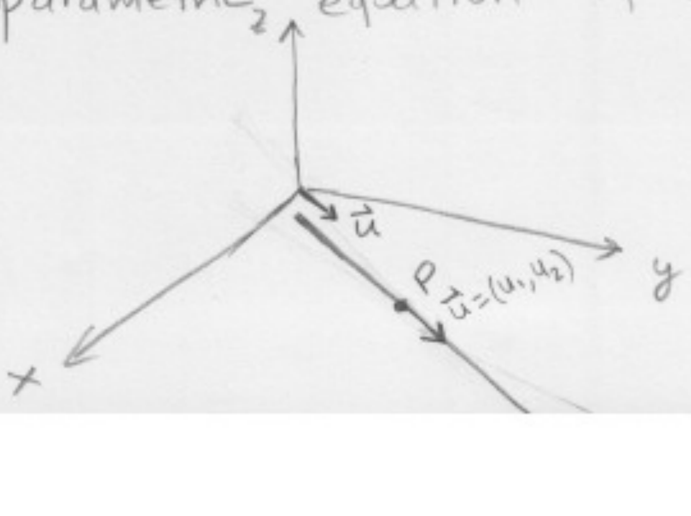
$$= \lim_{h\to 0} \frac{f(x_0+hu_1,\ y_0+hu_2) - f(x_0,y_0)}{h}$$

Note at $\vec{u}=(1,0)$, $\vec{u}=(0,1)$ correspond to $\frac{\partial f}{\partial x}(x_0,y_0)$ and $\frac{\partial f}{\partial y}(x_0,y_0)$ respectively, which are particular cases of partial derivatives.

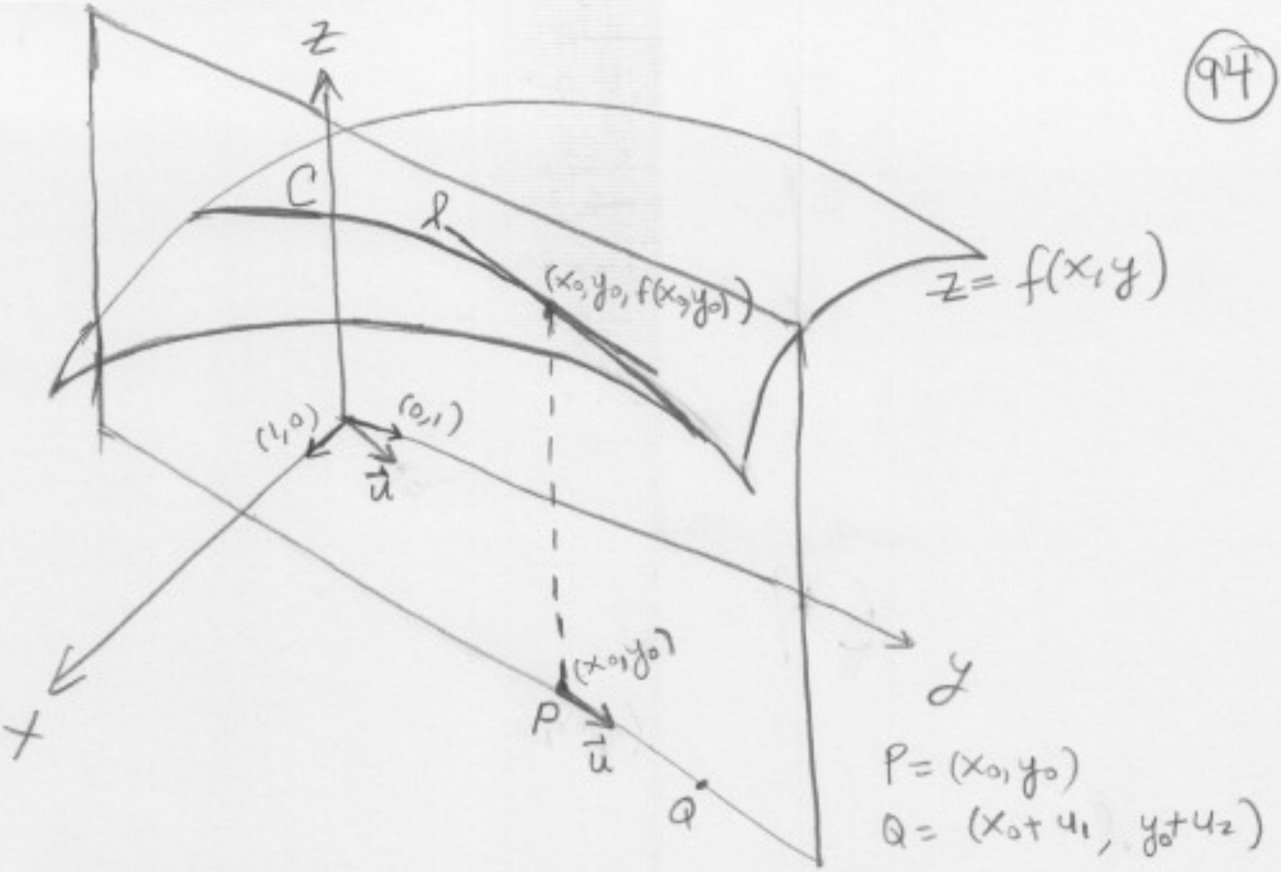
$$\frac{\partial f}{\partial x}(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h,\ y_0) - f(x_0,y_0)}{h}$$

$$\frac{\partial f}{\partial y}(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0,\ y_0+h) - f(x_0,y_0)}{h}$$

We define $\vec{r}(h) = (x_0+hu_1,\ y_0+hu_2)$. This is the parametric equation of a line, $\vec{r}(0) = (x_0,y_0)$.



$\vec{r}(h) = (x_0+hu_1,\ y_0+hu_2,\ 0)$
$P = (x_0, y_0, 0)$
We can embed the line in $\mathbb{R}^3$.



Look at the definition:

$$\frac{\partial f}{\partial \vec{u}}(x_0,y_0) = \lim_{h \to 0} \frac{f(x_0+hu_1, y+hu_2) - f(x_0,y_0)}{h}$$

$$= \lim_{h \to 0} \frac{f(\vec{r}(h)) - f(\vec{r}(0))}{h} = g'(0)$$

where $g = f \circ \vec{r} : \mathbb{R} \to \mathbb{R}$. Notice that $g(h)$ is evaluating f along the line $\vec{r}(h)$, and we are substracting $f(x_0,y_0)$, and dividing by h; ie, we are doing 1-dimensional calculus in the plane. Thus, $\frac{\partial f}{\partial \vec{u}}(x_0,y_0)$ is the slope of the tangent line l, at $(x_0, y_0, f(x_0,y_0))$, to the curve C which is the intersection of the graph $z=f(x,y)$ with the plane perpendicular to the x-y plane, that contains P and Q.

How to compute $\frac{\partial f}{\partial \vec{u}}(x_0, y_0)$ in practice?

Formula to compute $\frac{\partial f}{\partial \vec{u}}(x_0, y_0)$ <u>when f is differentiable</u> at $\vec{r}(0) = (x_0, y_0)$:

$$\frac{\partial f}{\partial \vec{u}}(x_0, y_0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h}, \quad g = f \circ \vec{r}$$
$$= g'(0).$$

Using the chain rule:

$g'(h) = (f \circ \vec{r})'(h) = \nabla f(\vec{r}(h)) \cdot \vec{r}\,'(h)$; the use of chain rule requires the f is differentiable at (x_0, y_0).

Letting $h = 0$:

$$g'(0) = \nabla f(\vec{r}(0)) \cdot \vec{r}\,'(0)$$
$$= \nabla f(x_0, y_0) \cdot (u_1, u_2), \quad \vec{r}\,'(h) = (u_1, u_2)$$

$$\therefore \quad \boxed{\frac{\partial f}{\partial \vec{u}}(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u}}$$

The previous arguments work in any dimension: if $f: \mathbb{R}^n \to \mathbb{R}$, f differentiable, then:

$$\boxed{\frac{\partial f}{\partial \vec{u}}(\vec{x}) = \nabla f(\vec{x}) \cdot \vec{u}}$$

<u>Other notation</u>: $D_{\vec{u}} f(x_0, y_0)$ instead of $\frac{\partial f}{\partial \vec{u}}(x_0, y_0)$.

Ex: Let $f(x,y,z) = x^2 y e^z$.

Find the directional derivative for f at $(1,0,0)$ in the direction of $\vec{u} = \frac{1}{\sqrt{14}}(1,2,3)$.

Since f is differentiable everywhere in $\mathbb{R}^3$, we can use the formula:

$$\frac{\partial f}{\partial \vec{u}}(1,0,0) = \nabla f(1,0,0) \cdot \frac{1}{\sqrt{14}}(1,2,3)$$

$$\nabla f = (2xye^z, x^2 e^z, x^2 y e^z)$$

$$= (0,1,0) \cdot \frac{1}{\sqrt{14}}(1,2,3) = \frac{2}{\sqrt{14}}$$

Remark: We have:

$$\nabla f(\vec{x}) \cdot \vec{u} = \|\nabla f\| \, \cancel{\|\vec{u}\|}^{\,1} \cos\theta$$

$$-1 \leq \cos\theta \leq 1$$

Then the max directional derivative occurs in the direction of $\vec{u} = \frac{\nabla f(\vec{x})}{\|\nabla f(\vec{x})\|}$. Indeed:

$$\nabla f(\vec{x}) \cdot \frac{\nabla f(\vec{x})}{\|\nabla f(\vec{x})\|} = \frac{\|\nabla f(\vec{x})\|^{\cancel{2}}}{\cancel{\|\nabla f(\vec{x})\|}} = \|\nabla f(\vec{x})\|$$

The min directional derivative occurs in the direction of $\vec{u} = -\frac{\nabla f(\vec{x})}{\|\nabla f(\vec{x})\|}$. Indeed:

$$-\nabla f(\vec{x}) \cdot \frac{\nabla f(\vec{x})}{\|\nabla f(\vec{x})\|} = -\frac{\|\nabla f(\vec{x})\|^{\cancel{2}}}{\cancel{\|\nabla f(\vec{x})\|}} = -\|\nabla f(\vec{x})\|$$