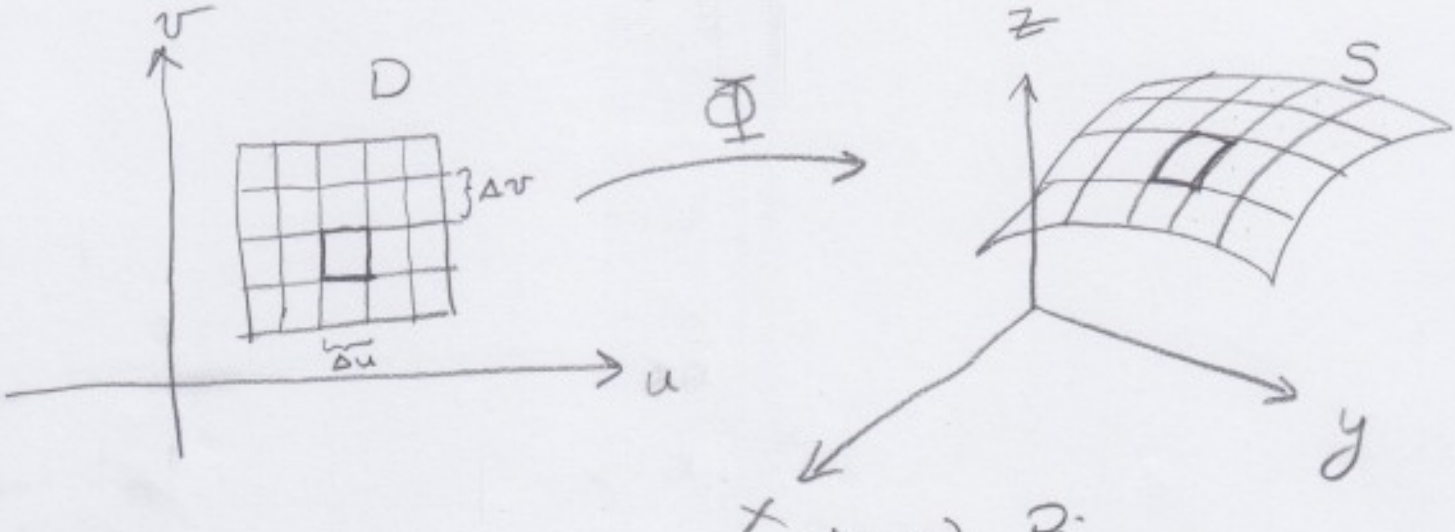
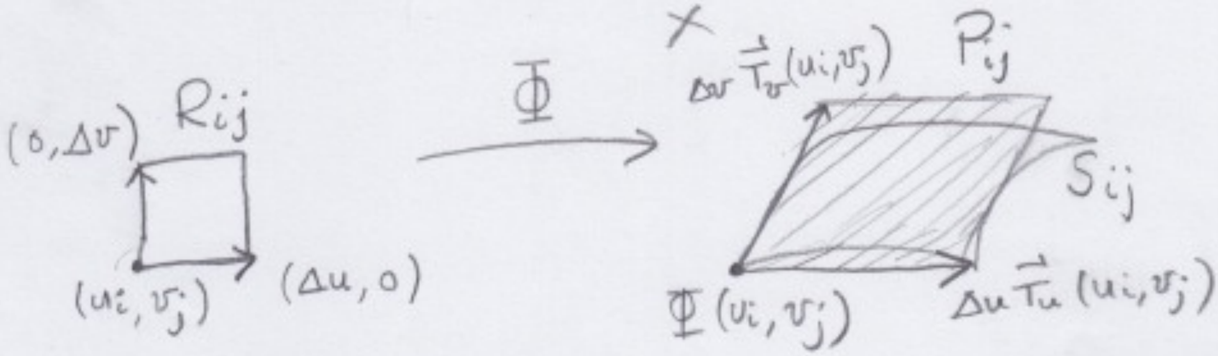
Section 7.4

Question is: How to compute the area of a surface.





$$\Phi(u,v) = (x(u,v),\ y(u,v),\ z(u,v))$$

$$D\Phi = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}$$

Evaluate $D\Phi\,(u_i,v_j)$

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} \begin{pmatrix} \Delta u \\ 0 \end{pmatrix} = \begin{pmatrix} \Delta u\,\frac{\partial x}{\partial u} \\ \Delta u\,\frac{\partial y}{\partial u} \\ \Delta u\,\frac{\partial z}{\partial u} \end{pmatrix} = \Delta u \begin{pmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial u} \end{pmatrix} = \Delta u\,\vec{T}_u$$

$$\underbrace{\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}}_{\substack{\parallel \\ D\Phi(u_i, v_j)}} \begin{pmatrix} 0 \\ \Delta v \end{pmatrix} = \begin{pmatrix} \Delta v \frac{\partial x}{\partial v} \\ \Delta v \frac{\partial y}{\partial v} \\ \Delta v \frac{\partial z}{\partial v} \end{pmatrix} = \Delta v \begin{pmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial v} \end{pmatrix}$$

$$= \Delta v \, \vec{T}_v$$

Recall

$$A(P) = \| \vec{a} \times \vec{b} \|$$

$$A(P_{ij}) = \| \Delta u \, \vec{T}_u(u_i, v_j) \times \Delta v \, \vec{T}_v(u_i, v_j) \|$$

$$= \| \vec{T}_u(u_i, v_j) \times \vec{T}_v(u_i, v_j) \| \, \Delta u \, \Delta v$$

$$A(S) \cong \sum_{i=1}^{n} \sum_{j=1}^{m} A(P_{ij}) \qquad \Phi \text{ is } C^1$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} \| \vec{T}_u(u_i, v_j) \times \vec{T}_v(u_i, v_j) \| \, \Delta u \, \Delta v$$

$$A(S) = \lim_{\substack{\Delta u \to 0 \\ \Delta v \to 0}} \sum_{i=1}^{n} \sum_{j=1}^{m} \| \vec{T}_u(u_i, v_j) \times \vec{T}_v(u_i, v_j) \| \, \Delta u \, \Delta v$$

$$= \iint_D \| \vec{T}_u(u, v) \times \vec{T}_v(u, v) \| \, du \, dv \qquad \blacksquare$$

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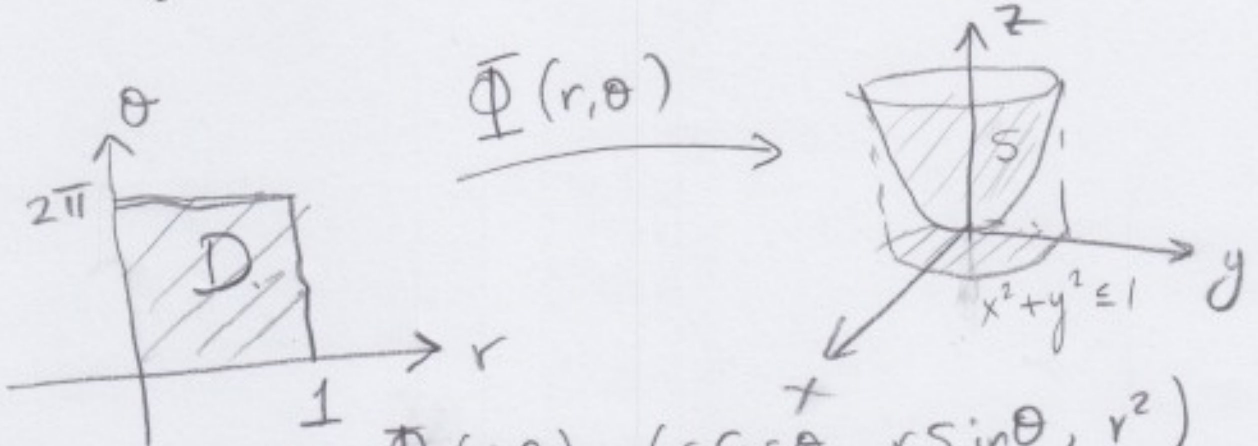
$$A(S) = \iint_D \|\vec{T}_u \times \vec{T}_v\| \, du\, dv.$$

<u>Ex</u>: Find the area of the surface of a paraboloid $z = x^2+y^2$ over the domain $x^2+y^2 \le 1$



$$\Phi(r,\theta) = (\underbrace{r\cos\theta}_{x}, \underbrace{r\sin\theta}_{y}, \underbrace{r^2}_{z})$$

$$\vec{T}_r = (\cos\theta, \sin\theta, 2r).$$

$$\vec{T}_\theta = (-r\sin\theta, r\cos\theta, 0)$$

$$A(S) = \iint_D \|\vec{T}_r \times \vec{T}_\theta\| \, dr\, d\theta$$

$$\vec{T}_r \times \vec{T}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= \vec{i}\,(-2r^2\cos\theta) - \vec{j}\,(2r^2\sin\theta) + \vec{k}(r)$$

$$= (-2r^2\cos\theta, -2r^2\sin\theta, r)$$

$$\|\vec{T}_r \times \vec{T}_\theta\| = \sqrt{4r^4 + r^2}$$

$$A(S) = \int_0^{2\pi} \int_0^1 \sqrt{4r^4 + r^2}\, dr\, d\theta$$

$\underset{r}{[0,1]} \times \underset{\theta}{[0,2\pi]}$

(we are not changing variables to polar coordinates)

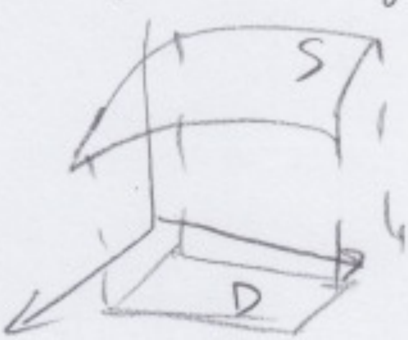
$$= \int_0^{2\pi} \int_0^1 r\sqrt{1+4r^2}\, dr\, d\theta$$

$$= 2\pi \, \frac{1}{8} \, \frac{(1+4r^2)^{3/2}}{3/2} \Bigg]_0^1 = \frac{\pi}{4} \cdot \frac{2}{3} \, (1+4r^2)^{3/2} \Bigg]_0^1$$

$$= \frac{\pi}{6}\left(5^{3/2} - 1\right)$$

<u>Ex</u>. Compute the area of the surface given by the function description

$$z = f(x,y), \quad (x,y) \in D.$$



We form a parametrization

$$\Phi : D \to \mathbb{R}^3$$

$$\Phi(u,v) = (u, v, f(u,v))$$

$$(u,v) \in D.$$

$$\vec{T}_u = \left(1, 0, \frac{\partial f}{\partial u}\right) \quad \vec{T}_v = \left(0, 1, \frac{\partial f}{\partial v}\right)$$

$$A(S) = \iint_D \|\vec{T}_u \times \vec{T}_v\|\, du\, dv.$$

$$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{\partial f}{\partial u} \\ 0 & 1 & \frac{\partial f}{\partial v} \end{vmatrix}$$

$$= \vec{i}\left(-\frac{\partial f}{\partial u}\right) - \vec{j}\left(\frac{\partial f}{\partial v}\right) + \vec{k}\,(1)$$

$$= \left(-\frac{\partial f}{\partial u}, -\frac{\partial f}{\partial v}, 1\right).$$

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2}\, du\, dv.$$

We re-do the problem with paraboloid using the function description formula

$$f(x,y) = x^2 + y^2 \quad \frac{\partial f}{\partial x} = 2x \quad \frac{\partial f}{\partial y} = 2y.$$

$$A(s) = \iint_{x^2+y^2 \le 1} \sqrt{1 + 4x^2 + 4y^2}\, dx\, dy$$

We <u>use</u> a change of variables to polar coordinates to obtain.

$$A(S) = \int_0^{2\pi}\int_0^1 \sqrt{1+4r^2} \cdot \underset{\text{Jacobian}}{r}\, dr\, d\theta.$$

And we are back to same integral as before.