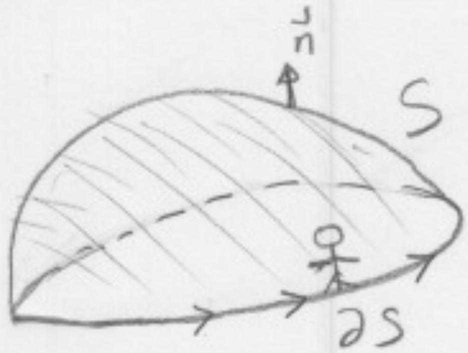
Section 8.2.

Stokes' theorem.

If S is given by a parametrization $\Phi : D \subset \mathbb{R}^2 \to S$ and ∂S is the curve forming the border of S (oriented positively), then:

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS = \int_{\partial S} \vec{F} \cdot d\vec{r}$$



Positive orientation means that S is to the left as the curve is traversed.

<u>Ex</u>: Let $\vec{F} = (x^2+y-4,\ 3xy,\ 2xz+z^2)$ and S is the surface $x^2+y^2+z^2=16$.

Then $\displaystyle\iint_S (\nabla\times\vec{F})\cdot\vec{n}\,dS = 0$ since there is no boundary curve. But if S is the upper half of the sphere $x^2+y^2+z^2=16$, $z \geq 0$. then we have:

$\vec{r}(t) = (4\cos t, 4\sin t, 0)$
$0 \leq t \leq 2\pi$

$$\iint_S (\nabla\times\vec{F})\cdot\vec{n}\,dS = \int_{\partial S}\vec{F}\cdot d\vec{r}.$$

$$= \int_0^{2\pi} \vec{F}(\vec{r}(t))\cdot\frac{\vec{r}\,'(t)}{\cancel{\|\vec{r}\,'(t)\|}}\cancel{\|\vec{r}\,'(t)\|}\,dt$$

$$= \int_0^{2\pi}(16\cos^2 t + 4\sin t - 4,\ \underline{3(4\cos t)(4\sin t)},\ 0)\cdot(-4\sin t,\ \underline{4\cos t},\ 0)\,dt.$$

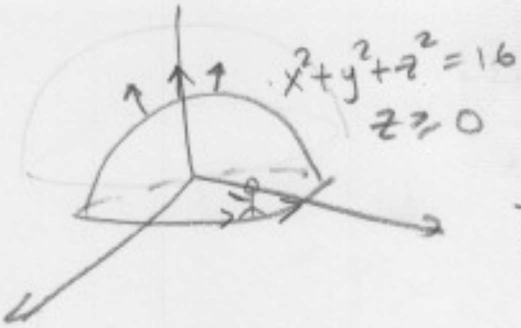
$$= \int_0^{2\pi}(192\cos^2 t\sin t - 48\cos^2 t\sin t - 16\sin^2 t + 16\sin t)\,dt$$

$$= \int_0^{2\pi}(128\cos^2 t\sin t - 16\underbrace{\sin^2 t}_{\frac{1}{2}(1-\cos 2t)} + 16\sin t)\,dt$$

$$= \left[-\frac{128}{3}\cos^3 t - \frac{16}{2}t + \frac{16}{4}\sin 2t - 16\cos t\right]_0^{2\pi}$$

$$= \cancel{\frac{-128}{3}} - 16\pi \cancel{-16} + \cancel{\frac{128}{3}} + \cancel{16} = -16\pi$$

Solution 2: Compute directly.



$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, dS$$

$$\Phi(x,y) = (x, y, \sqrt{16-x^2-y^2})$$

$$(x,y) \in D \quad D = x^2+y^2 \leq 16$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y-4 & 3xy & 2xz+z^2 \end{vmatrix} = -\vec{j}\,(2z) + \vec{k}\,(3y-1)$$

$$= (0, -2z, 3y-1).$$

$$\vec{T}_x \times \vec{T}_y = \left(\frac{-\partial z}{\partial x}, \frac{-\partial z}{\partial y}, 1 \right) \qquad \frac{\partial z}{\partial x} = \frac{-x}{\sqrt{16-x^2-y^2}} \quad \frac{\partial z}{\partial y} = \frac{-y}{\sqrt{16-x^2-y^2}}$$

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, \underline{\underline{dS}} = \iint_{x^2+y^2 \leq 4} (\nabla \times \vec{F})(\Phi(x,y)) \cdot \frac{\vec{T}_x \times \vec{T}_y}{\cancel{\|\vec{T}_x \times \vec{T}_y\|}} \cancel{\|\vec{T}_x \times \vec{T}_y\|}\, dx\, dy$$

$$= \iint_{x^2+y^2 \leq 16} (0, -2\sqrt{16-x^2-y^2}, 3y-1) \cdot \left(\frac{x}{\sqrt{16-x^2-y^2}}, \frac{y}{\sqrt{16-x^2-y^2}}, 1 \right) dx\,dy$$

$$= \iint_{x^2+y^2 \leq 16} (-2y + 3y - 1)\, dx\, dy = \iint_{x^2+y^2 \leq 16} (y-1)\, dx\, dy.$$

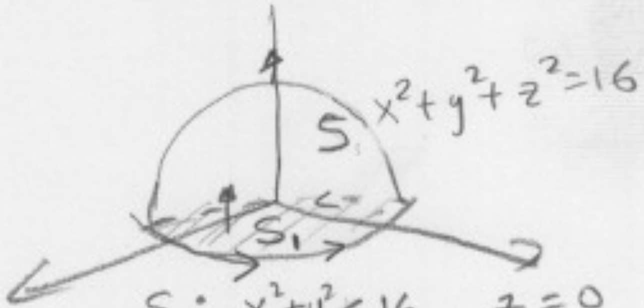
$$= \int_0^{2\pi} \int_0^4 (r \sin\theta - 1)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^4 (r^2 \sin\theta - r)\, dr\, d\theta = \int_0^{2\pi} \left[\frac{r^3}{3} \sin\theta - \frac{r^2}{2} \right]_0^4 d\theta$$

$$= \int_0^{2\pi} \left(\frac{64}{3} \sin\theta - 8 \right) d\theta = \left[-\frac{64}{3} \cos\theta - 8\theta \right]_0^{2\pi} = \cancel{\frac{-64}{3}} - 16\pi + \cancel{\frac{64}{3}}$$

$$= -16\pi \quad \blacksquare$$

Solution # 3.



$S_1 : x^2+y^2 \le 16 , \ z=0$

Both S and S_1 have the same boundary.

$$\iint_S (\nabla\times\vec{F})\cdot\vec{n}\, dS = \int_{\partial S} \vec{F}\cdot d\vec{r}$$

$$\iint_{S_1} (\nabla\times\vec{F})\cdot\vec{n}\, dS = \int_{\partial S} \vec{F}\cdot d\vec{r}$$

$$\Rightarrow \iint_S (\nabla\times\vec{F})\cdot\vec{n}\, dS = \iint_{S_1} (\nabla\times\vec{F})\cdot\vec{n}\, dS$$

$S_1 : \ \Phi(x,y) = (x,y,0) , \ (x,y)\in D , \ D = x^2+y^2 \le 16 .$

$\vec{T}_x \times \vec{T}_y = (0,0,1)$

$$\iint_{S_1} (\nabla\times\vec{F})\cdot\vec{n}\, dS = \iint_{x^2+y^2=16} (0,0,3y-1)\cdot(0,0,1)\, dx\, dy$$

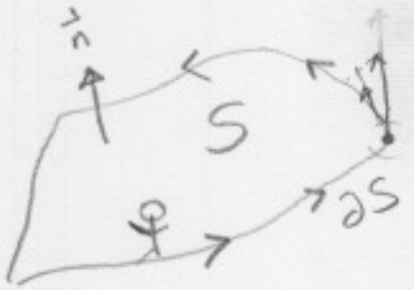
$$= \iint_{x^2+y^2\le 16} (3y-1)\, dx\, dy = \iint_{x^2+y^2\le 16} 3y \ - 16\pi$$

$$= \int_0^{2\pi}\int_0^4 3r\sin\theta \cdot r\, dr\, d\theta \ - 16\pi$$

$$= 3\int_0^{2\pi} \frac{64}{3}\cdot \sin\theta\, d\theta \ - 16\pi$$

$$= 64\left[\cancel{-\cos\theta}^{\,0}\right]_0^{2\pi} - 16\pi = -16\pi \quad \blacksquare$$

Physical meaning of curl of a vector field.



$$\iint_S (\nabla\times F)\cdot\vec{n}\, dS = \int_{\partial S}\vec{F}\cdot d\vec{r}$$

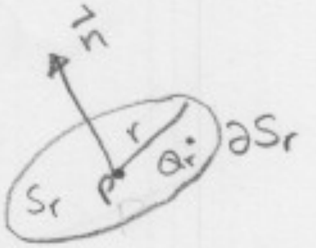
$$\int_{\partial S}\vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}\,\underline{\|\vec{r}\,'(t)\|\, dt}$$

$$\| $$

$$\int_{\partial S}\vec{F}\cdot\vec{T}\,\underline{ds}$$

Fix idea: Let $\vec{V}$ be the velocity vector field of a fluid.

Put inside fluid a paddle wheel.



$$\iint_{S_r}(\text{Curl}\,\vec{V})\cdot\vec{n}\, dS = \int_{\partial S_r}\vec{V}\cdot\vec{T}\, ds.$$

$\|$ mean value theorem

$$\exists Q_r \quad (\nabla\times\vec{V})(Q_r)\cdot\vec{n}\, A(S_r)$$

$$(\nabla\times\vec{V})(Q_r)\cdot\vec{n} = \frac{1}{A(S_r)}\int_{\partial S_r}\vec{V}\cdot\vec{T}\, ds$$

$r\to 0$.

$$\lim_{r\to 0} (\nabla\times\vec{v})(Q_r)\cdot\vec{n} = \lim_{r\to 0} \frac{1}{A(S_r)}\int_{\partial S_r} \vec{v}\cdot\vec{T}\,ds$$

$$\boxed{(\nabla\times\vec{v})(P)\cdot\vec{n} = \lim_{r\to 0} \frac{1}{A(S_r)}\int_{\partial S_r} \vec{v}\cdot\vec{T}\,ds}$$

$\int_{\partial S_r} \vec{v}\cdot\vec{T}\,ds =$ Net amount of turning of the fluid

or "circulation of $\vec{v}$ around C".

$(\nabla\times\vec{v})(P)\cdot\vec{n}$ turning or rotating effect of the fluid around the axis $\vec{n}$.

$(\nabla\times\vec{v})(P)\cdot\vec{n}$ = Circulation of $\vec{v}$ per unit area at P on the surface perpendicular to $\vec{n}$.

$(\nabla\times\vec{v})(P)\cdot\vec{n}$ is maximized when $\vec{n} = \dfrac{(\nabla\times\vec{v})(P)}{\|(\nabla\times\vec{v})(P)\|}$. Therefore the rotating effect at P is greatest about the axis parallel to $\dfrac{\text{Curl}\,\vec{v}}{\|\text{Curl}\,\vec{v}\|}$.

$\Rightarrow$ Curl $\vec{v}$ is called the vorticity vector

* A vector field $\vec{F}$ with $\nabla\times\vec{F} = 0$ is called irrotational.

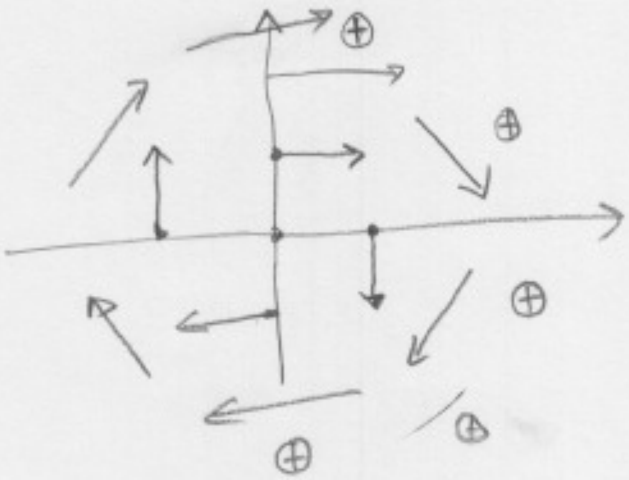
Ex: Let $\vec{v}(x,y,z) = \left(\frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2}\right)$.

$\vec{v}$ is irrotational since

$$\nabla \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{y}{x^2+y^2} & \frac{-x}{x^2+y^2} & 0 \end{vmatrix}$$

$$= \vec{i}\,(0) - \vec{j}\,(0) + \vec{k}\left[\frac{\partial}{\partial x}\left(\frac{-x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right]$$

$$= \vec{0} = (0,0,0)$$



$(1,0) \to (0,-1)$
$(0,1) \to (1,0)$
$(-1,0) \to (0,1)$
$(0,-1) \to (-1,0)$

The fluid moves in circles around the origin, but being irrotational means that "looking at a paddle wheel from above a moving fluid, the paddle wheel <u>does not</u> rotate arount its axis". ∎