Section 8.3

Ex: Faraday's Law.

Vector Calculus → Electromagnetism

$\vec{E}$: Time-dependant electric vector field, $\vec{E}(x,y,z,t)$.

$\vec{H}$: Time-dependant magnetic field, $\vec{H}(x,y,z,t)$.

$\int_C \vec{E}\cdot d\vec{r}$ = Voltage around C.

$\iint_S \vec{H}\cdot\vec{n}\, dS$ = magnetic flux accross S.

Remember other notations:

$$\int_C \vec{E}\cdot d\vec{r} \quad \text{or} \quad \int_C \vec{E}\cdot\vec{T}\, ds$$

$$\iint_S \vec{H}\cdot\vec{n}\, dS \quad \text{or} \quad \iint_S \vec{H}\cdot d\vec{S}$$

Faraday's law: Voltage around C equals the negative rate of change of magnetic flux through S:

Ex: Show that Faraday's law follows from one of Maxwell's equations.

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t}\vec{H} \quad \leftarrow \text{Maxwell eq.}$$

Answer: $$\int_C \vec{E} \cdot d\vec{r} = \iint_S (\nabla \times \vec{E}) \cdot \vec{n} \, dS$$

$$= -\iint_S \frac{\partial \vec{H}}{\partial t} \cdot \vec{n} \, dS$$

$$= -\frac{\partial}{\partial t} \iint_S \vec{H} \cdot \vec{n} \, dS.$$

$$\int_C \vec{E} \cdot d\vec{r} = -\frac{\partial}{\partial t} \iint_S \vec{H} \cdot \vec{n} \, dS$$

Faraday's law

Recall:

$\vec{F}$ is a gradient vector field (or conservative vector field) if

$$\vec{F} = \nabla f$$

for some function f, called a potential for $\vec{F}$.

Ex: $V = \dfrac{GmM}{r}$, $r = (x^2+y^2+z^2)^{1/2}$

$V : \mathbb{R}^3 \to \mathbb{R}$ Gravitational potential energy.

$$\vec{F} = \nabla V$$

$$\vec{F} = -\frac{GmM}{r^3}(x, y, z).$$

$$\frac{\partial V}{\partial x} = \frac{\partial}{\partial x}\left[GmM\,(x^2+y^2+z^2)^{-1/2} \right]$$

$$= -\frac{GmM}{\cancel{2}}\,(x^2+y^2+z^2)^{-3/2}(\cancel{2}x)$$

$$= -GmM\,\frac{x}{r^3}$$

$$\frac{\partial V}{\partial x} = -GmM\,\frac{y}{r^3} \qquad \frac{\partial V}{\partial z} = -GmM\,\frac{z}{r^3}$$

$$\nabla V = -GmM\left(\frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3}\right) = \vec{F}$$

$$p \iff q$$

<u>Thm</u>: $\vec{F} = \nabla f$ <u>if and only if</u> $\nabla \times \vec{F} = \vec{0}$.

If the exam, we showed that:

* If $\vec{F} = \nabla f$ then $\nabla \times \vec{F} = \vec{0}$.

$$p \Rightarrow q$$

* For proof of $q \Rightarrow p$, <u>see book</u>.

<u>Ex</u>: Consider

$$\vec{F}(x,y,z) = (y,\ z\cos yz + x,\ y\cos yz).$$

Show that $\vec{F}$ is irrotational and find a scalar potential for $\vec{F}$.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z\cos yz + x & y\cos yz \end{vmatrix}$$

$$= \vec{i}\left(\cancel{\cos yz} - \cancel{yz\sin yz} - \cancel{\cos yz} + \cancel{zy\sin yz}\right)$$

$$- \vec{j}\,(0 - 0) + \vec{k}\,(1 \cancel{-} 1)$$

$$= (0, 0, 0) = \vec{0}.$$

$\vec{F}(x,y,z) = (y,\ z\cos yz + x,\ y\cos yz)$.

By Thm, $\exists f: \mathbb{R}^3 \to \mathbb{R}$ s.t.

$$\vec{F} = \nabla f.$$

$$\frac{\partial f}{\partial x} = y \qquad \frac{\partial f}{\partial y} = z\cos yz + x \qquad \frac{\partial f}{\partial z} = y\cos yz.$$

$$f(x,y,z) = \int y\,dx = xy + h_1(y,z)$$

$$\frac{\partial f}{\partial x} = y + 0 \checkmark$$

$$\frac{\partial f}{\partial y} = \cancel{x} + \frac{\partial h_1}{\partial y} = \cancel{x} + z\cos yz$$

$$= \frac{\partial h_1}{\partial y} = z\cos yz$$

$$h_1(y,z) = \int z\cos yz\,dy$$

$$= \sin yz + h_2(z)$$

$$f(x,y,z) = xy + \sin yz + h_2(z).$$

$$\frac{\partial f}{\partial y} = x + z\cos yz + 0 \checkmark$$

$$\frac{\partial f}{\partial z} = y\cos yz + h_2'(z) = y\cos yz$$

$$h_2'(z) = 0, \quad h_2(z) = C.$$

$$\boxed{f(x,y,z) = xy + \sin yz + C.} \checkmark$$