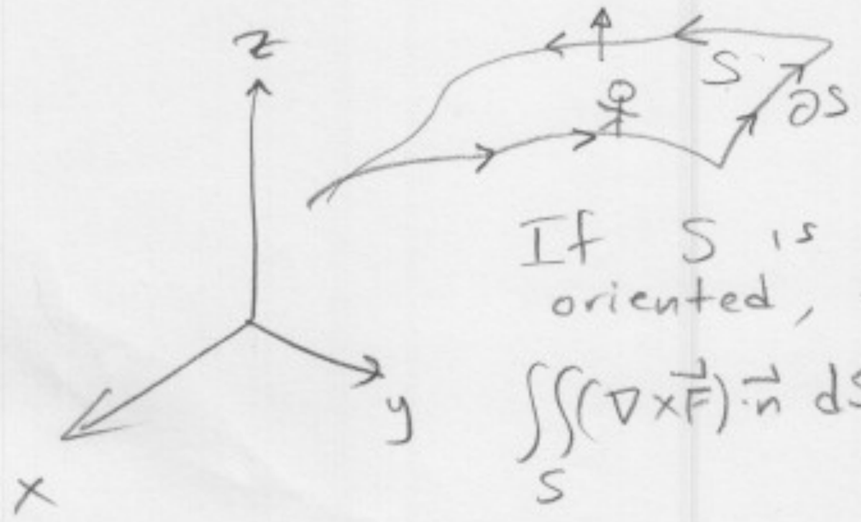
Section 8.1

Green's theorem.

Green's theorem is just Stokes' theorem in the plane; so it is a particular case of Stokes' theorem.

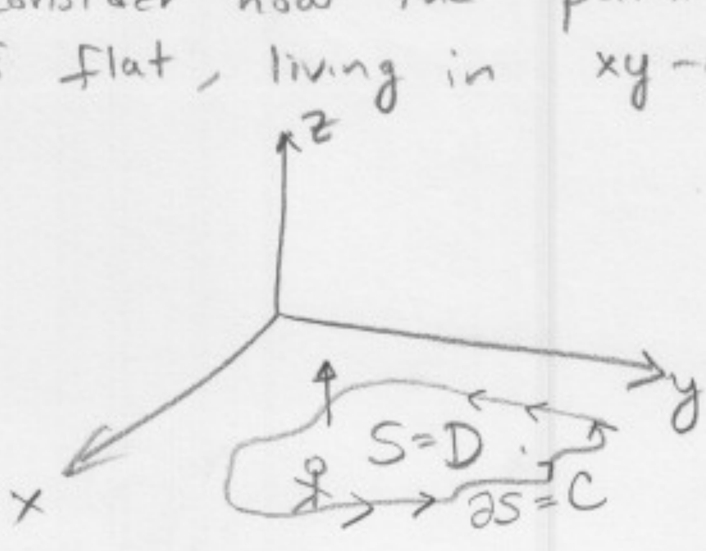
Recall Stokes' theorem



If S is positively oriented, then.

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS = \int_{\partial S} \vec{F} \cdot d\vec{r}.$$

Consider now the particular case when S is flat, living in xy-plane.



$S = D$ is trivially parametrized as $\Phi(x,y) = (x,y,0)$ $(x,y) \in D$.

$\vec{T}_x \times \vec{T}_y = (0,0,1)$.

Consider a vector field that only depends on x,y variables:

$$\vec{F} = (P(x,y), \underline{Q(x,y)}, 0).$$

In particular, Stokes' theorem holds in D.

$$\boxed{\iint_D (\nabla\times\vec{F})\cdot\vec{n}\, dS = \int_C \vec{F}\cdot d\vec{r}}$$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = \vec{i}\,(0) - \vec{j}\,(0) + \vec{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$\iint_D \nabla\times\vec{F}\cdot\vec{n}\,\underline{\underline{dS}} = \iint_D \left(0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\cdot \frac{\vec{T}_x\times\vec{T}_y}{\cancel{\|\vec{T}_x\times\vec{T}_y\|}}\, \cancel{\|\vec{T}_x\times\vec{T}_y\|}\, dx\,dy$$

$$= \iint_D \left(0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\cdot(0,0,1)\, dx\, dy.$$

$$= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy.$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\vec{T}\, \underline{\underline{ds}}$$

$\vec{r}(t) = (x(t), y(t), 0)$
$a \le t \le b.$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}\,'(t)}{\cancel{\|\vec{r}\,'(t)\|}} \cancel{\|\vec{r}\,'(t)\|}\, dt$$

$$= \int_a^b (P, Q, 0) \cdot (x'(t), y'(t), 0)\, dt$$

$$= \int_a^b \left[P(x(t), y(t))\, x'(t) + Q(x(t), y(t))\, y'(t)\right] dt$$

$$\iint_D (\nabla \times \vec{F})\cdot\vec{n}\, dS = \int_C \vec{F}\cdot d\vec{r}$$

$$\qquad \| \qquad\qquad\qquad\qquad \|$$

$$\boxed{\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dx\,dy = \int_a^b \left[P\, x'(t) + Q\, y'(t)\right] dt}$$

Green's theorem.

Another notation:

$$\int_C P\, dx := \int_a^b P(x(t), y(t))\, x'(t)\, dt$$

$$\int_C Q\, dy := \int_a^b Q(x(t), y(t))\, y'(t)\, dt$$

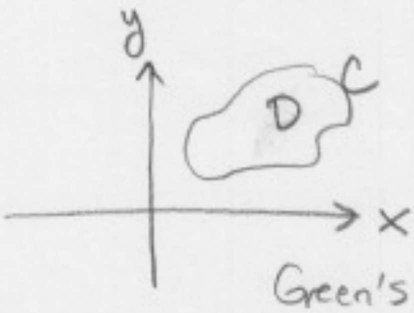
With this notation, we often see the Gauss–Green theorem stated as:

$$\boxed{\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_C P\,dx + Q\,dy.}$$

Ex: Green's theorem gives a formula to compute the area inside a region D. Let C be the boundary of D oriented positively. Show that the area of D is:

$$A = \frac{1}{2}\int_C x\,dy - y\,dx$$

Form the vector field $\vec{F} = (P, Q)$
$= (-y, x)$.



Green's theorem says:

$$\frac{1}{2}\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \frac{1}{2}\int_C P\,dx + Q\,dy$$

$$= \frac{1}{2}\int_C -y\,dx + x\,dy.$$

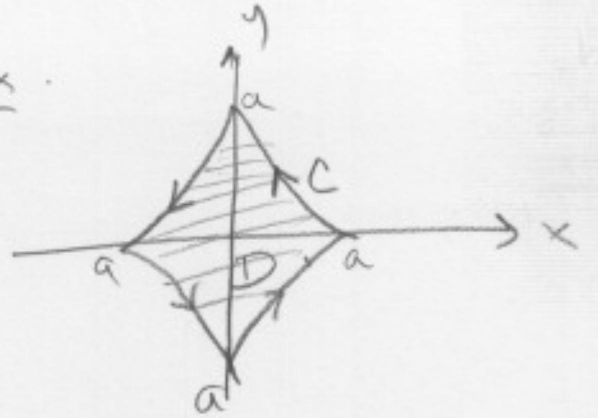
$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_D 1 - (-1)\,dx\,dy$$

$$= 2\iint_D dx\,dy = 2A(D).$$

$$\frac{1}{\cancel{2}}\left(\cancel{2}A(D)\right) = \frac{1}{2}\int_C -y\,dx + x\,dy$$

$$\boxed{A(D) = \frac{1}{2}\int_C x\,dy - y\,dx}$$

Ex.



hypocycloid. C

$$x^{2/3} + y^{2/3} = a^{2/3}$$

We can parametrize C as follows:

$$x(\theta) = a\cos^3\theta$$
$$y(\theta) = a\sin^3\theta, \quad 0 \le \theta \le 2\pi.$$

Compute $A(D)$.

$$A(D) = \frac{1}{2}\int_C x\,dy - y\,dx.$$

$$= \frac{1}{2}\int_0^{2\pi}\Big[(a\cos^3\theta)(3a\sin^2\theta\cos\theta) - (a\sin^3\theta)(-3a\cos^2\theta\sin\theta)\Big]\,d\theta$$

$$= \frac{3a^2}{2}\int_0^{2\pi}(\cos^4\theta\sin^2\theta + \cos^2\theta\sin^4\theta)\,d\theta.$$

$$= \frac{3a^2}{2}\int_0^{2\pi}\sin^2\theta\cos^2\theta\,d\theta.$$

$$= \frac{3a^2}{8}\int_0^{2\pi}\sin^2 2\theta\,d\theta = \frac{3a^2}{8}\int_0^{2\pi}\frac{1}{2}(1-\cos 4\theta)\,d\theta$$

$$= \boxed{\frac{3a^2\pi}{8}}$$