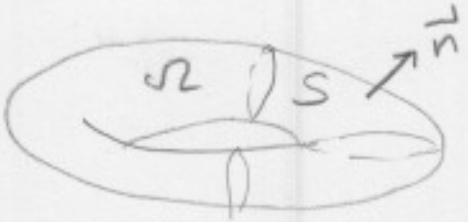
Section 8.4.

Gauss' Theorem

(Divergence Theorem).

Suppose that a surface S encloses a volume Ω. Let $\vec{n}$ be the outer normal vector field on S.

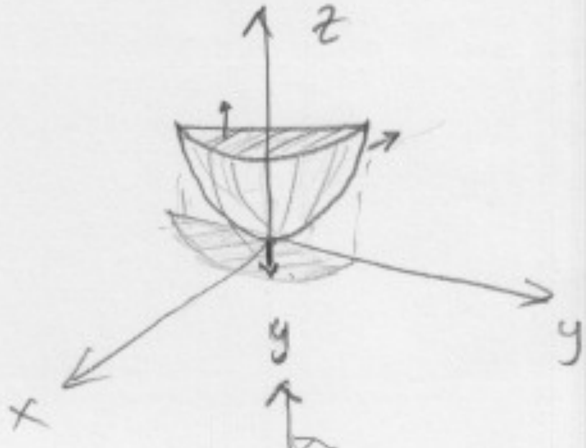


$\partial\Omega = S$

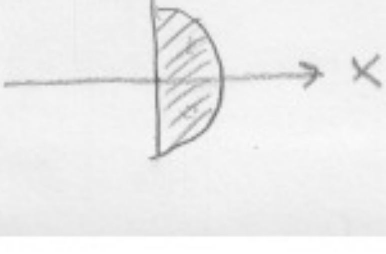
Then, if $\vec{F}$ is a C^1 vector field on Ω. and S, then

$$\boxed{\iint_S \vec{F}\cdot\vec{n}\, dS = \iiint_\Omega \operatorname{div}\vec{F}\, dx\,dy\,dz}$$

Ex: Let $\vec{F} = (y, z, xz)$ and Ω be the set $x^2+y^2 \le z \le 1$, $x \ge 0$.



Evaluate $\iint_{\partial\Omega} F\cdot\vec{n}\, dS$ where $\partial\Omega$ is the whole boundary of Ω with outward normal.



$$\iint_{\partial\Omega} \vec{F}\cdot\vec{n}\, dS = \iint_{\text{Top}} \vec{F}\cdot\vec{n}\, dS + \iint_{\text{Back}} \vec{F}\cdot\vec{n}\, dS + \iint_{\text{Front}} \vec{F}\cdot\vec{n}\, dS.$$

① Top: $\Phi(x,y) = (x, y, 1)$,

$$(x,y) \in D = x^2+y^2 \le 1,\ x \ge 0.$$

$$\vec{T}_x \times \vec{T}_y = (0,0,1).$$

$$\iint_{\substack{x^2+y^2\le 1\\ x\ge 0}} (y, 1, x)\cdot(0,0,1)\, dx\,dy = \iint_{\substack{x^2+y^2\le 1\\ x\ge 0}} x\, dx\,dy.$$

$$= \int_{-\pi/2}^{\pi/2}\int_0^1 (r\cos\theta)\, r\, dr\, d\theta = \frac{1}{3}\left[\sin\theta\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{2}{3}.$$

② <u>Back</u>: $\Phi(y,z) = (0, y, z)$

$$y^2 \le z \le 1 \quad \vec{T}_y \times \vec{T}_z = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \vec{i}\,(1) - \vec{j}\,(0) + \vec{k}\,(0)$$

$$= (1,0,0).$$

$$\iint_{y^2\le z\le 1} (y, z, 0)\cdot \frac{\vec{T}_z\times\vec{T}_y}{\cancel{\|\vec{T}_z\times\vec{T}_y\|}}\ \cancel{\|\vec{T}_z\times\vec{T}_y\|}\, dz\,dy$$

$$= \iint_{y^2\le z\le 1} (y,z,0)\cdot(-1,0,0)\, dz\,dy = \int_{-1}^{1}\int_{y^2}^{1} -y\, dz\, dy$$

$$= \int_{-1}^{1} -y\,[z]_{y^2}^{1} = \int_{-1}^{1}(-y+y^3)\, dy = 0$$

(3) Front: $\Phi(x,y) = (x, y, \overset{f(x,y)}{\overset{\shortparallel}{x^2+y^2}})$.

$$\vec{T}_x \times \vec{T}_y = \left(\frac{-\partial f}{\partial x}, \frac{-\partial f}{\partial y}, 1\right).$$

$$= (-2x, -2y, 1)$$

$$\iint_{Back} \vec{F}\cdot\vec{n}\, dS = \iint_{\substack{x^2+y^2\leq 1 \\ x\geq 0}} (y, x^2+y^2, x(x^2+y^2)) \cdot \frac{\vec{T}_y \times \vec{T}_x}{\cancel{\|\vec{T}_y\times\vec{T}_x\|}} \cancel{\|\vec{T}_y\times\vec{T}_x\|}\, dx\,dy$$

$$= \iint_{\substack{x^2+y^2\leq 1 \\ x\geq 0}} (y, x^2+y^2, x(x^2+y^2)) \cdot (2x, 2y, -1)\, dx\, dy.$$

$$= \iint_{\substack{x^2+y^2\leq 1 \\ x\geq 0}} \left[2xy + 2y(x^2+y^2) - x(x^2+y^2)\right] dx\, dy$$

$$= -\int_{-\pi/2}^{\pi/2}\int_0^1 (r\cos\theta) r^2 \cdot \underline{\underline{r}}\; dr\, d\theta.$$

$$= -\int_{-\pi/2}^{\pi/2} \frac{1}{5}\cos\theta\, d\theta = -\frac{1}{5}\left[\sin\theta\right]_{-\pi/2}^{\pi/2} = -\frac{2}{5}$$

$$\therefore \iint_{\partial\Omega} \vec{F}\cdot\vec{n}\, dS = \frac{2}{3} + 0 - \frac{2}{5} = \frac{10-6}{15} = \frac{4}{15}$$

Second Solution.

Use divergence Theorem:

$$\iint_{\partial\Omega} \vec{F} \cdot \vec{n}\, dS = \iiint_{\Omega} \operatorname{div} \vec{F}\, dx\,dy\,dz.$$

$$\vec{F} = (y, z, xz)$$

$$\operatorname{div} \vec{F} = 0 + 0 + x = x$$

$$= \iiint_{\Omega} x \, dx\,dy\,dz.$$

$$= \int_{-\pi/2}^{\pi/2} \int_0^1 \int_{r^2}^1 (r\cos\theta)\cdot r \, \underline{dz}.\,dr.\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \int_0^1 r^2 \cos\theta \,(1-r^2)\, dr\, d\theta$$

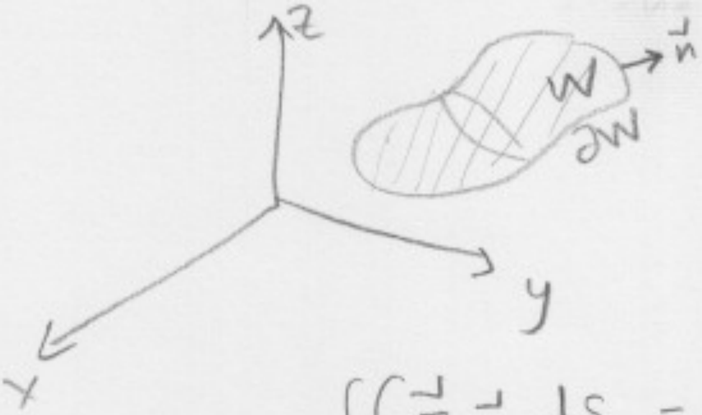
$$= \int_{-\pi/2}^{\pi/2} \int_0^1 r^2 \underline{\cos\theta}\, dr\, d\theta - \int_{-\pi/2}^{\pi/2} \int_0^1 r^4 \underline{\cos\theta}\, dr\, d\theta$$

$$= \frac{1}{3} \left[\sin\theta\right]_{-\pi/2}^{\pi/2} - \frac{1}{5}\left[\sin\theta\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{2}{3} - \frac{2}{5} = \frac{10-6}{15} = \frac{4}{15}$$ Same answer.

Section 8.4, continuation

Divergence theorem

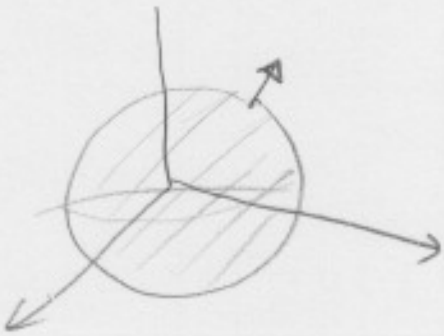


$\vec{n}$ outer normal.

$$\iint_{\partial W} \vec{F}\cdot\vec{n}\, dS = \iiint_{W} \operatorname{div}\vec{F}\, dx\,dy\,dz$$

<u>Ex</u>: Let $\vec{F} = (x^3, y^3, z^3)$ and S the sphere of radius 1, $\vec{n}$ outer normal. Compute

$$\iint_{S} \vec{F}\cdot\vec{n}\, dS.$$



$$\iint_{S} \vec{F}\cdot\vec{n}\, dS = \iiint_{x^2+y^2+z^2\le 1} \operatorname{div}\vec{F}\, dx\,dy\,dz.$$

$$= \iiint_{x^2+y^2+z^2\le 1} (3x^2+3y^2+3z^2)\, dx\,dy\,dz.$$

$$= 3\int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho^2\,\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta.$$

$$= \frac{3}{5}(2\pi)\left[\int_0^{\pi} \sin\varphi\, d\varphi\right] = \frac{6\pi}{5}\left[-\cos\varphi\right]_0^{\pi} = \frac{6\pi}{5}(1-(-1)) = \frac{12\pi}{5}$$

Physical meaning of divergence.

We now use the Gauss' theorem to show that $\operatorname{div} \vec{v}$ gives a measure of the compressibility of the fluid.

Let P be a point in the domain Ω of $\vec{v}$, and B_r a ball of radius r centered at P in Ω. Let ∂B_r denote the boundary of B_r and $\vec{n}$ the outward normal on ∂B_r

$$\iint_{\partial B_r} \vec{v}\cdot\vec{n}\, dS = \iiint_{B_r} \operatorname{div} \vec{v} \underset{MVT}{=} (\operatorname{div} \vec{v})(Q_r)\, V(B_r).$$

$$\exists Q_r$$

$$\operatorname{div} \vec{v}\,(Q_r) = \frac{1}{V(B_r)} \iint_{\partial B_r} \vec{v}\cdot\vec{n}\, dS.$$

let $r \to 0$.

$$\lim_{r\to 0} \operatorname{div} \vec{v}\,(Q_r) = \lim_{r\to 0} \frac{1}{V(B_r)} \iint_{\partial B_r} \vec{v}\cdot\vec{n}\, dS$$

$$\operatorname{div} \vec{v}\,(P) = \lim_{r\to 0} \frac{1}{V(B_r)} \iint_{\partial B_r} \vec{v}\cdot\vec{n}\, dS$$

Thus, if $\operatorname{div} \vec{v}\,(P) > 0$ we say that P is a source and if $\operatorname{div} \vec{v}(P) < 0$ it is a sink.

If $\operatorname{div} \vec{v} = 0$, we say that the fluid is incompressible and there are no sources and sinks.