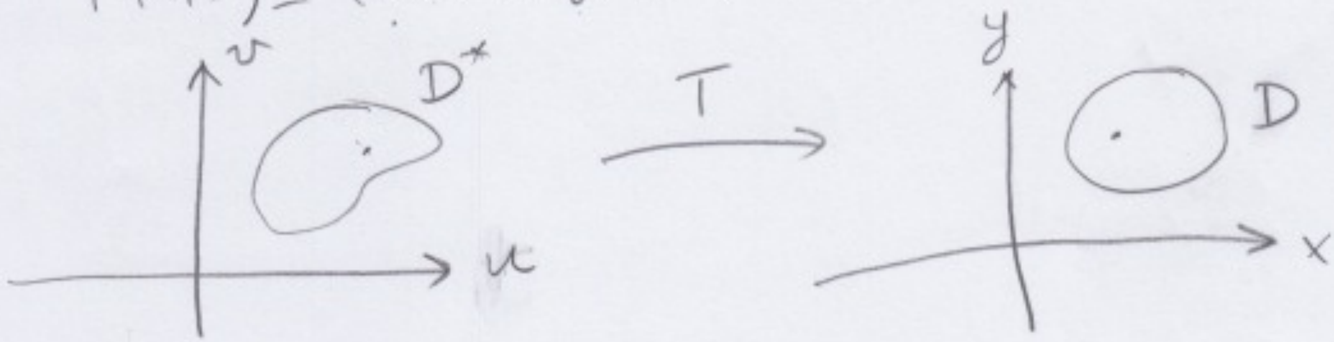
Change of variables formula

<u>Theorem</u> : suppose f is continuous on D^* and $T : D^* \to D$, T has continuous partial derivatives (i.e, T is C'). Then:

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \underline{du\,dv}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \det(DT) \qquad \begin{aligned} T : D^* &\to D \\ \mathbb{R}^2 &\to \mathbb{R}^2 \end{aligned}$$

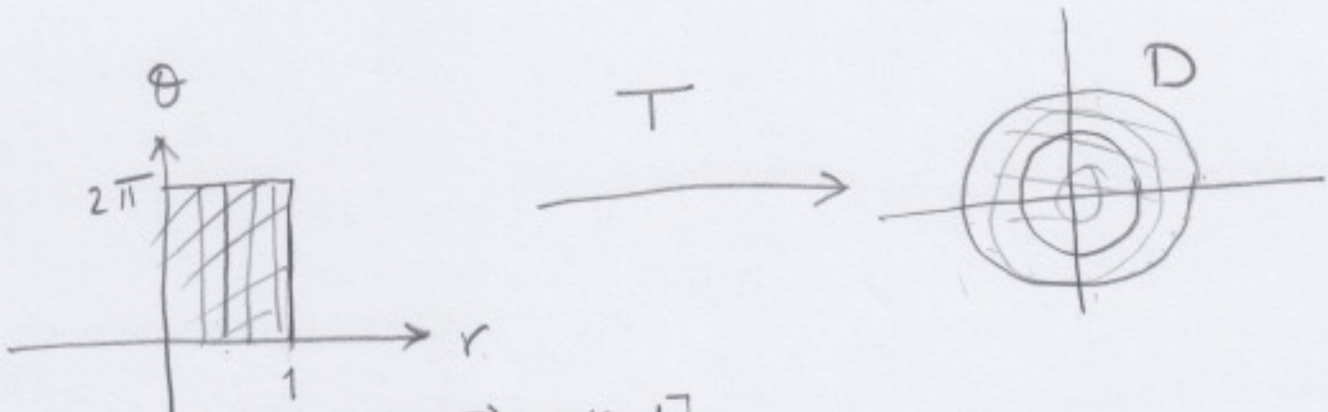
$$T(u,v) = (x(u,v), y(u,v))$$



$$\underset{2\times 2}{DT} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Ex: $\displaystyle\iint_D \sqrt{1+x^2+y^2}\, dx\,dy$

D unit disk $\{x^2+y^2 \le 1\}$.



$[0, 2\pi) \times (0, 1]$

$$T(r,\theta) = (\underbrace{r\cos\theta}_{x}, \underbrace{r\sin\theta}_{y})$$
$$T(1,\theta) = (\cos\theta, \sin\theta)$$

T is one-to-one map

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \det(DT)$$
$$DT = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$\det(DT) = r\cos^2\theta + r\sin^2\theta = r$$

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = r$$

$$\iint_D \sqrt{1+x^2+y^2}\, dx\, dy =$$

$$\int_0^{2\pi}\int_0^1 \sqrt{1+(r\cos\theta)^2+(r\sin\theta)^2}\cdot r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{1+r^2}\cdot r\, dr\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}\left[\frac{(1+r^2)^{3/2}}{3/2}\right]_0^1 d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\left[(1+r^2)^{3/2}\right]_0^1 d\theta$$

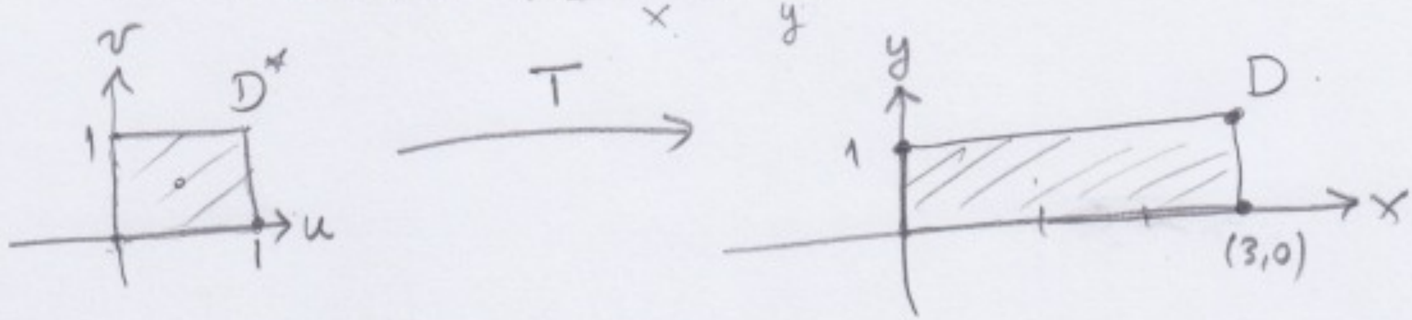
$$= \frac{1}{3}\int_0^{2\pi}(2^{3/2}-1)\, d\theta$$

$$= \frac{2\pi}{3}(2^{3/2}-1)$$

Ex: $\displaystyle\iint_D (x+y^2)\,dx\,dy$.

where D is the image $T(D^*)$, D^* is the unit square $[0,1]\times[0,1]$.

$$T(u,v) = (\underbrace{-u^2+4u}_{x}, \underbrace{v}_{y})$$



$$T(1,0) = (3,0)$$
$$T(0,1) = (0,1)$$
$$T(1,1) = (3,1)$$

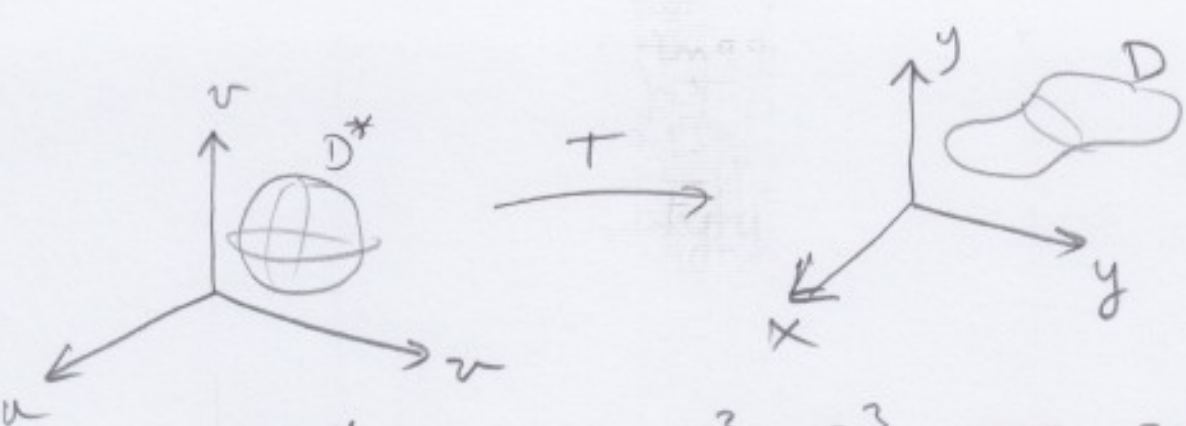
$$\overset{I}{\overset{\shortparallel}{\iint_D (x+y^2)\,dx\,dy}} = \int_0^1\int_0^1 (-u^2+4u+v^2)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \det(DT) = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{pmatrix} = \det\begin{pmatrix} -2u+4 & 0 \\ 0 & 1\end{pmatrix}$$

$$= -2u+4$$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = 4-2u$$

$$I = \int_0^1\int_0^1 (-u^2+4u+v^2)(4-2u)\,du\,dv$$ ... finish.

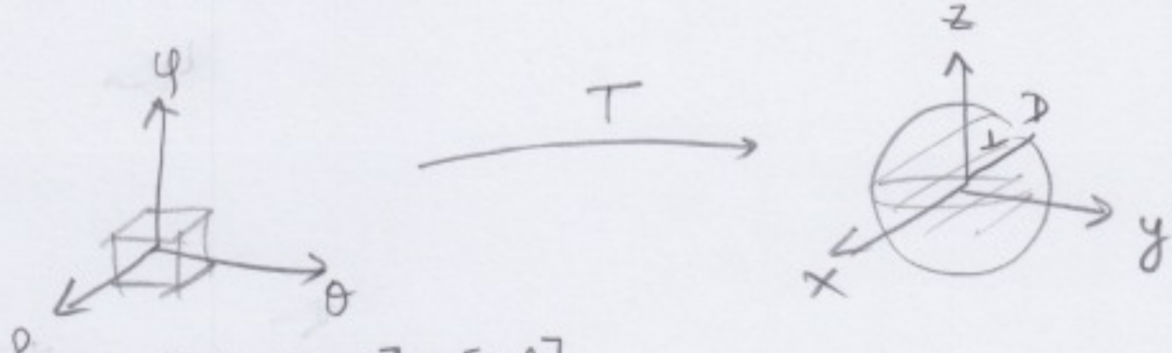


$$T : D^* \to D \qquad T : \mathbb{R}^3 \to \mathbb{R}^3 \qquad DT \quad 3\times 3$$

$$T(u,v,w) = (x(u,v,w),\ y(u,v,w),\ z(u,v,w)).$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det(DT) = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}$$

$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_{D^*} f(x(u,v,w),\ y(u,v,w),\ z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$



$$[0,1]\times[0,2\pi]\times[0,\varphi]$$

$$T(\rho,\theta,\varphi) = (\underbrace{\rho\sin\varphi\cos\theta}_{x},\ \underbrace{\rho\sin\varphi\sin\theta}_{y},\ \underbrace{\rho\cos\varphi}_{z})$$

Compute $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$

$$DT = \begin{pmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta & \rho\cos\varphi\sin\theta \\ \cos\varphi & 0 & -\rho\sin\varphi \end{pmatrix}$$

$$\det(DT) = \cos\varphi\left(-\rho^2\sin^2\theta\sin\varphi\cos\varphi - \rho^2\cos^2\theta\sin\varphi\cos\varphi\right)$$

$$-\rho\sin\varphi\left(\rho\sin^2\varphi\cos^2\theta + \rho\sin^2\varphi\sin^2\theta\right)$$

$$= -\rho^2\cos\varphi\sin\varphi\cos\varphi - \rho^2\sin\varphi(\sin^2\varphi)$$

$$= -\rho^2\sin\varphi(\cos^2\varphi + \sin^2\varphi) = -\rho^2\sin\varphi$$

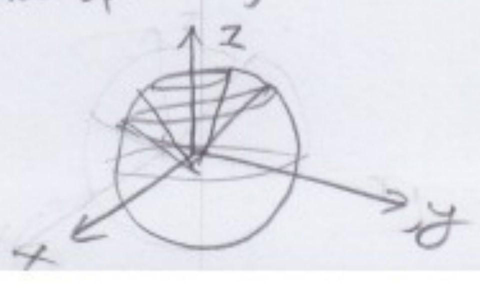
$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \rho^2\sin\varphi$$

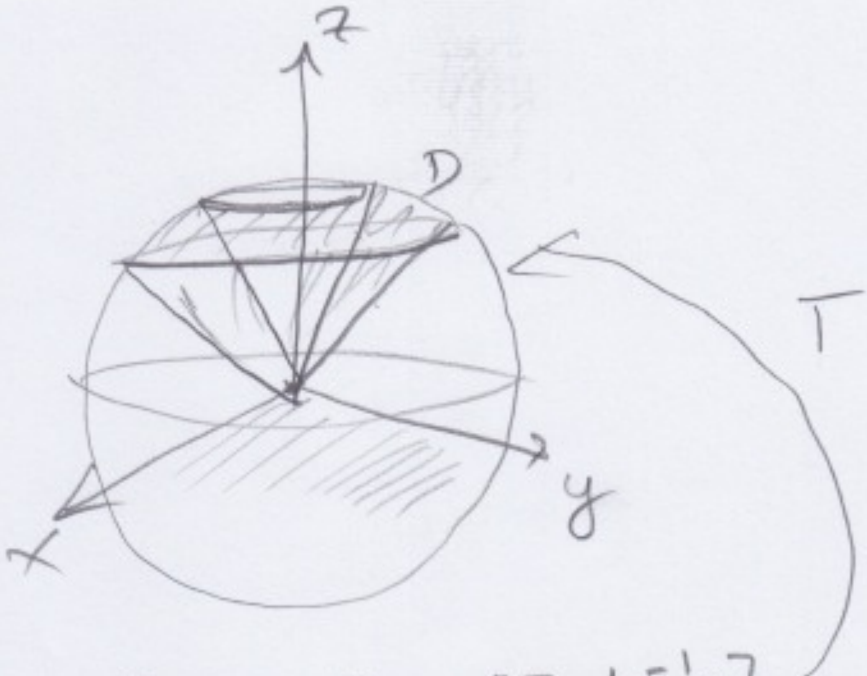
$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_{D^*} f(\rho\sin\varphi\cos\theta,\ \rho\sin\varphi\sin\theta,\ \rho\cos\varphi)\rho^2\sin\varphi\ d\rho\,d\theta\,d\varphi.$$

$x^2+y^2+z^2 \le 1$

$$\int_0^{\pi}\int_0^{2\pi}\int_0^1$$

$\iiint_D f(x,y,z)\,dx\,dy\,dz$, where D is the region in the first octant bounded by cones $\varphi = \frac{\pi}{4}$, $\varphi = \tan^{-1} 2$ and the sphere $\rho = \sqrt{6}$





$$D^* = \underset{\rho}{[0, \sqrt{6}]} \times \underset{\theta}{[0, \tfrac{\pi}{2}]} \times \underset{\varphi}{[\tfrac{\pi}{4}, \tan^{-1} 2]}$$

$$\iiint_D \underset{\sqrt{x^2+y^2+z^2}}{f} \, dV = \int_{\pi/4}^{\tan^{-1} 2} \int_0^{\pi/2} \int_0^{\sqrt{6}} \underset{f}{\rho} \cdot \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi .$$

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