Section 2.1

The geometry of real valued functions:

Although we will usually stick to functions of 2 or 3 variables, multivariable calculus applies to functions defined in n-dimensional space $\mathbb{R}^n$.

Specifying a function requires

(i) A formula for computation

(ii) A description of the set of values $U \subset \mathbb{R}^n$ which will be substituted in to the formula. U is called the domain of the function:

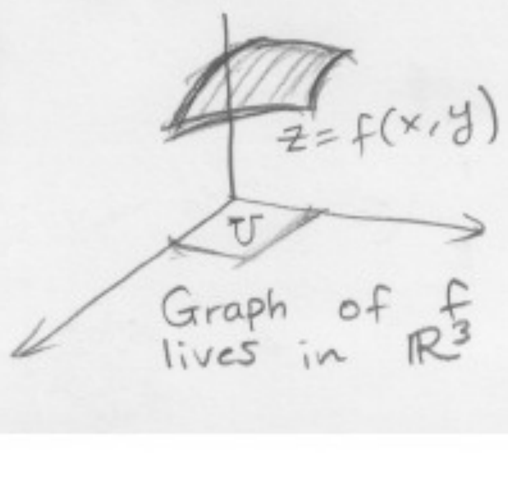
Notation: $f: U \subset \mathbb{R}^2 \to \mathbb{R}$ (graph lives in $\mathbb{R}^3$)

$$U = (0,1)\times(0,1)$$

$$f(x,y) = x+y$$

$f: \mathbb{R}^3 \to \mathbb{R}$ (graph lives in $\mathbb{R}^4$)

$$f(x,y,z) = x+y+z^2 \quad (U = \mathbb{R}^3)$$



In general, we use the notation:

$$f: U \subset \mathbb{R}^n \to \mathbb{R}$$

Level curves

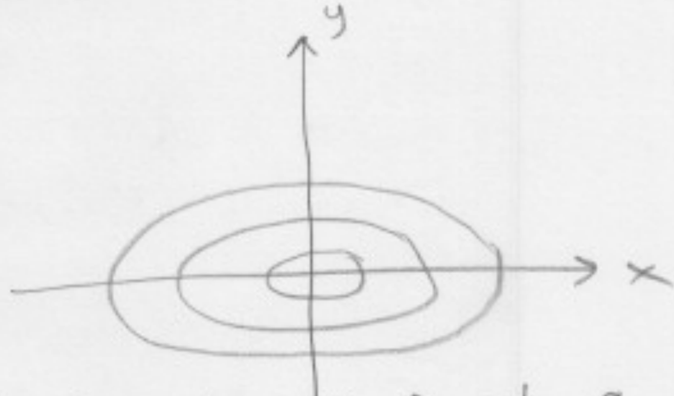
Consider the function $f: \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = x^2 + 4y^2$, or

$(x, y) \longmapsto x^2 + 4y^2$

Definition: The level curve of f at c is the set of all points (x,y) such that $f(x,y) = c$

Values of c	Level set
$c = 2 \longrightarrow$	Ellipse $x^2 + 4y^2 = 2$
$c = 1 \longrightarrow$	Ellipse $x^2 + 4y^2 = 1$
$c = 0 \longrightarrow$	$x^2 + y^2 = 0$, only $(0,0)$
$c = -2 \longrightarrow$	No level set

Level curves of f live in $\mathbb{R}^2$. The graph of f lives in $\mathbb{R}^3$.



The level curve of f at c is the projection on to the plane x-y of the intersection of the graph of f with the plane $z = c$.

Sections:

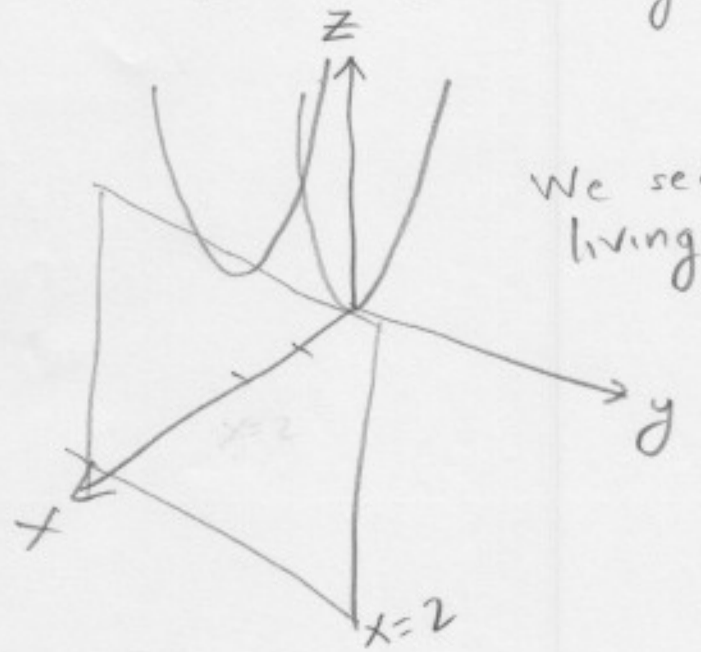
A section is a slice of a graph of $f(x,y)$ by a vertical plane:

Ex: $f(x,y) = x^2 + 4y^2$

- We first intersect the graph with vertical planes $x = c$

Value of c

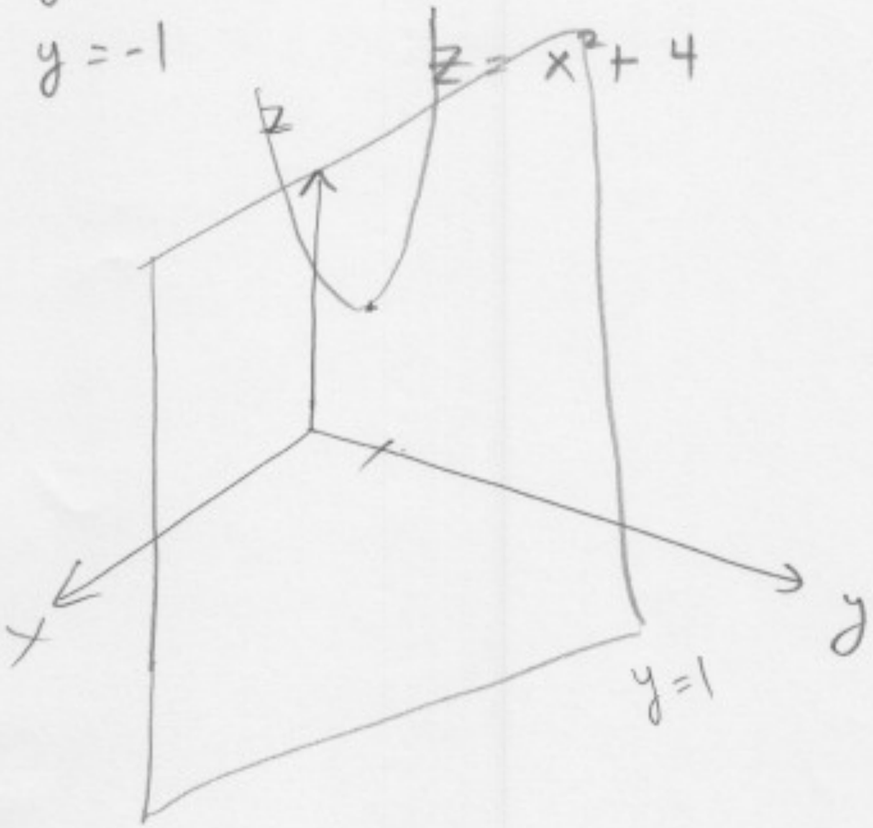
$x=2$	$z = 4 + 4y^2$
$x=0$	$z = 4y^2$
$x=-2$	$z = 4 + 4y^2$

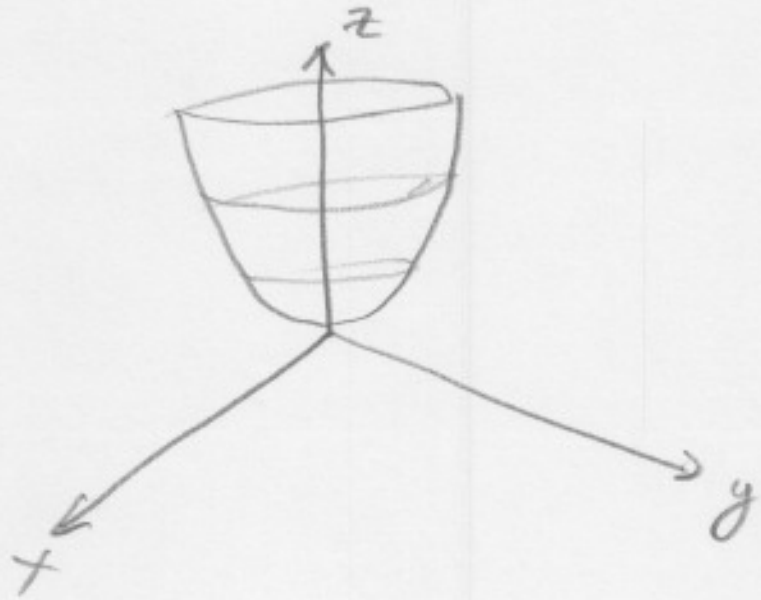


We see parabolas living in the planes $x = c$

- We now intersect the graph with vertical planes $y = c$.

Vale of c	
$y=1$	$z = x^2+4$
$y=0$	$z = x^2$
$y=-1$	$z = x^2+4$





Ex: $f(x,y) = x^2 - y^2$

$z = x^2 - y^2$

- Level curves

$z = c$

$x^2 - y^2 = c$

Hyperbolas

$z = 0$

$x^2 - y^2 = 0$

$y = \pm\sqrt{x^2 - c}$

- Sections:

$y = c$

$z = x^2 - c^2$

$y = 0$

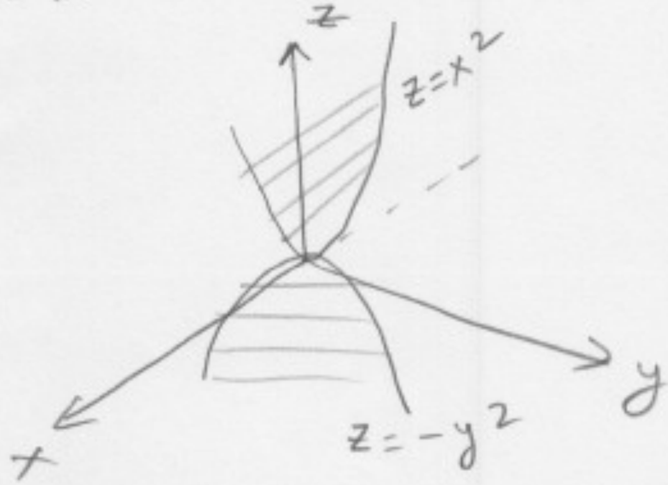
$z = x^2$

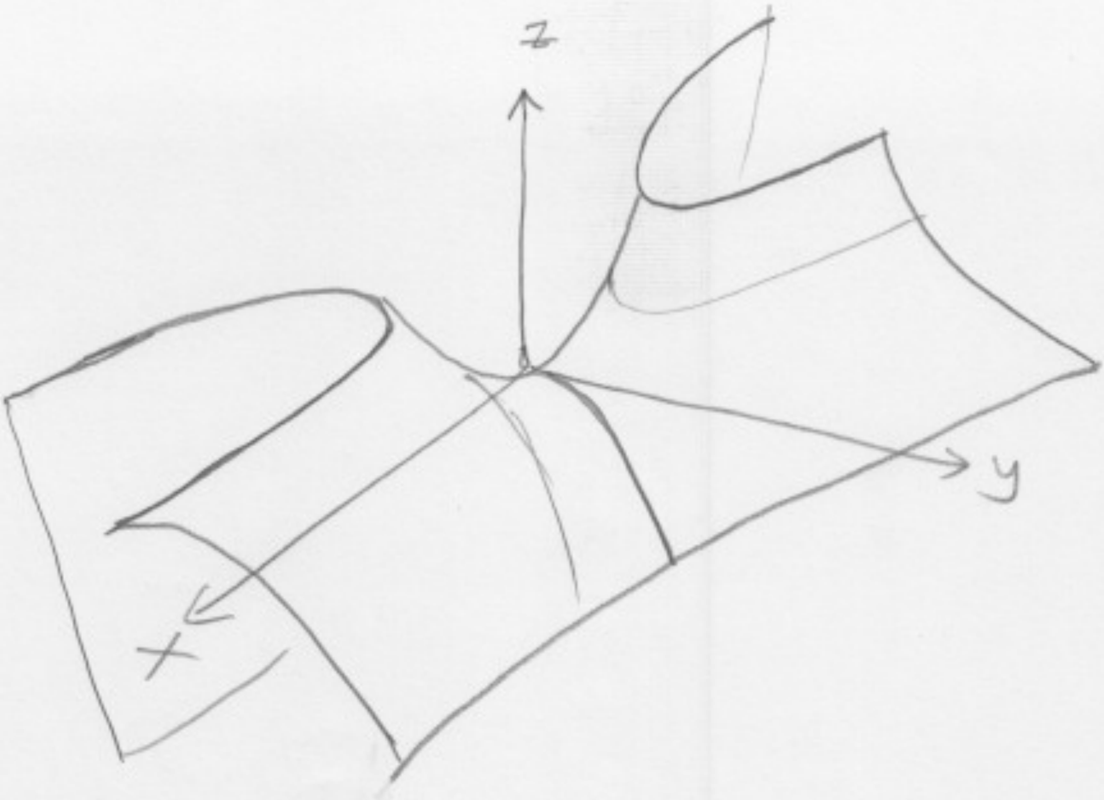
$x = c$

$z = c^2 - y^2$

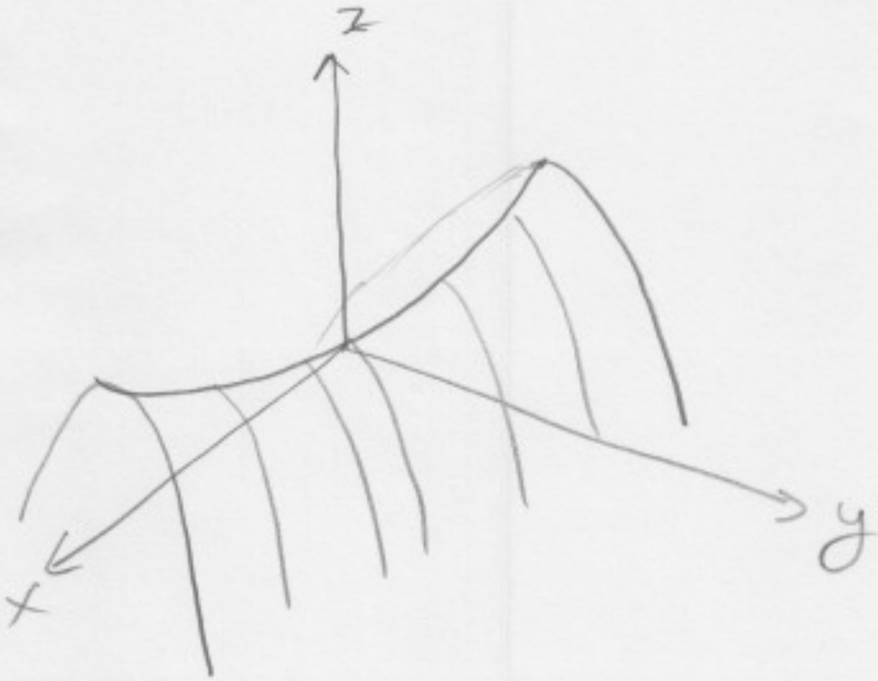
$c = 0$

$z = -y^2$





Another view:



Level surfaces

For functions of 3 variables $f: \mathbb{R}^3 \to \mathbb{R}$ we talk about level surfaces instead of level curves

Ex: $f(x,y,z) = x^2+z^2-y^2$

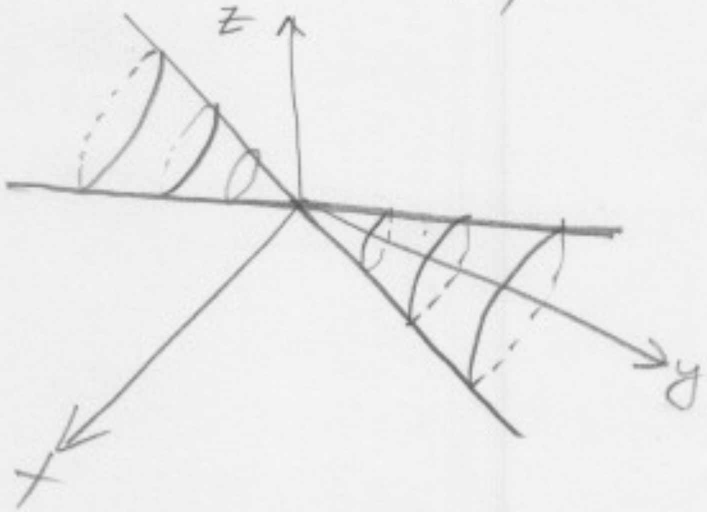
The level surface of f at c is the set of all points (x,y,z) such that $f(x,y,z)=c$

The level surfaces of f are:

$$x^2+z^2-y^2=c$$

For $c=0$, we have the surface:

$$x^2+z^2=y^2,$$



$x^2+z^2=y^2$, or $y = \pm\sqrt{x^2+z^2}$ is a cone with two branches

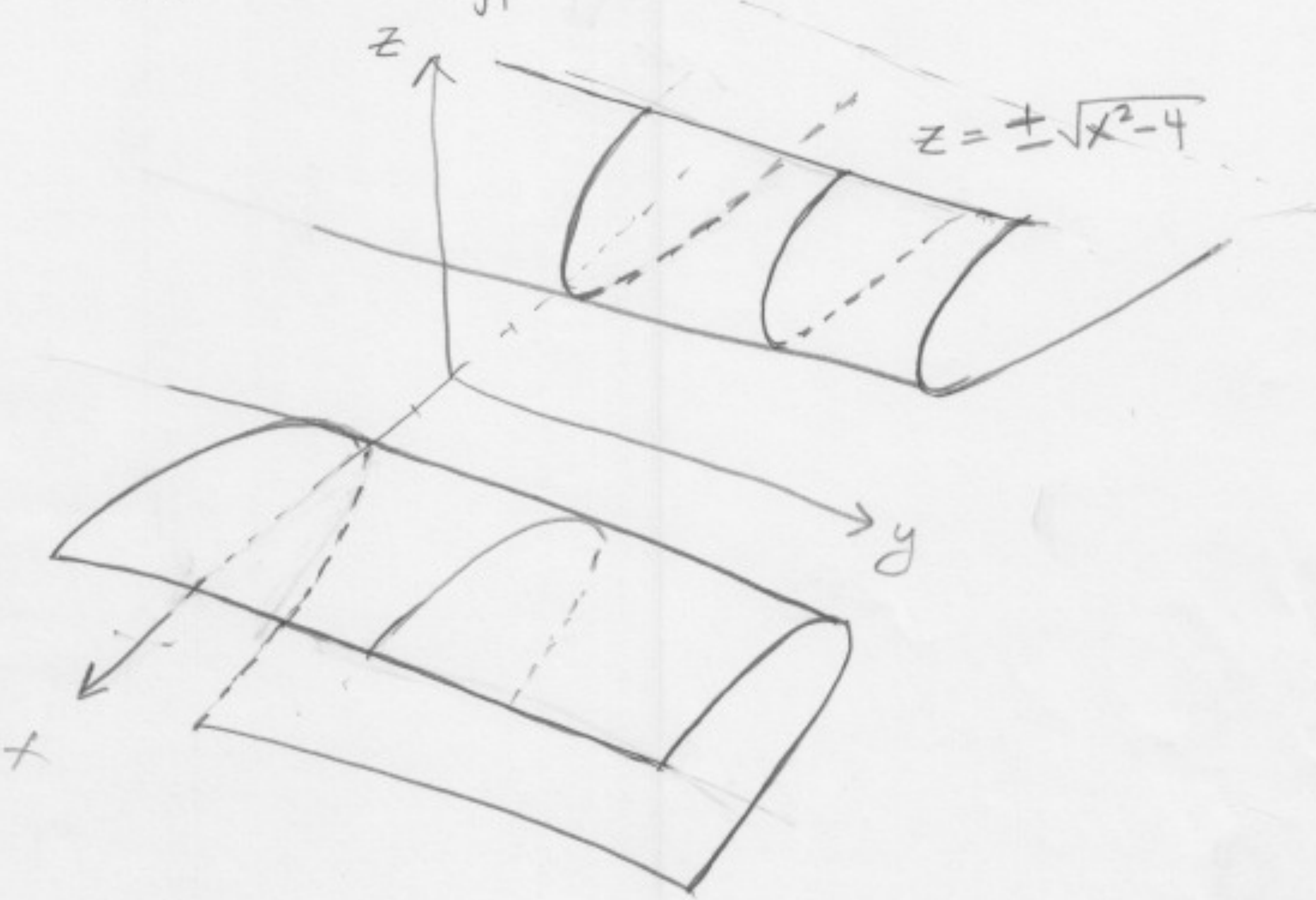
There is no reason why all variables need to appear.

Ex: Let $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = 4-x^2+z^2$.
The level surfaces $f(x,y,z) = c$ are "cylinders" (not circular cylinder)

For the case $c=0$ we have the surface $4-x^2+z^2=0$

or $x^2-z^2=4$,

which is a hyperbola.



The surface $x^2-z^2=4$ lives in $\mathbb{R}^3$, even though the variable y does not appear in the equation.

The surface is a cylinder, and all the sections are hyperbolas.

The fact the y is not is the formula means that the surface extends along the "y" axis, i.e, any point (x, y, z) belongs to the surface as long as $x^2-z^2=4$.