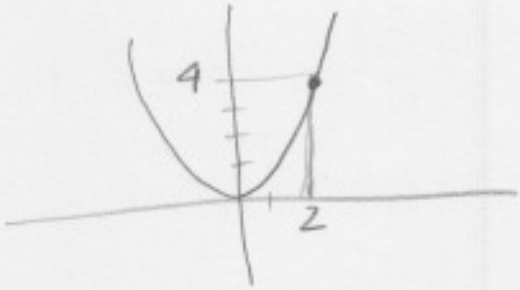
Section 2.2
Limits & Continuity

Recall the concept of limit from functions of one variable.

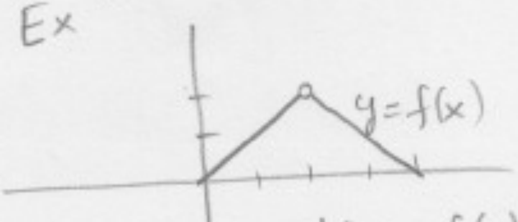
$$\lim_{x\to 2} f(x) = 4, \quad f(x)=x^2$$



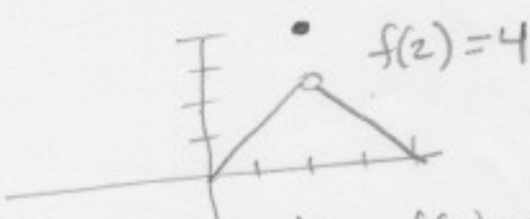
$\lim_{x\to x_0} f(x) = \ell$ means that "as x gets closer and closer to x_0, the value of $f(x)$ gets closer and closer to ℓ"

A function might not be defined at the limiting value, but the limit still exits:

Ex



$\lim_{x\to 2} f(x) = 2$



$\lim_{x\to 2} f(x) = 2$

In the second picture, $f(2)$ is defined but $\lim_{x\to 2} f(x) = 2 \neq f(2)$; thus the limit exists even though $f(2)$ takes a different value.

In 1-dimension, recall L'Hopital rule to compute limits:

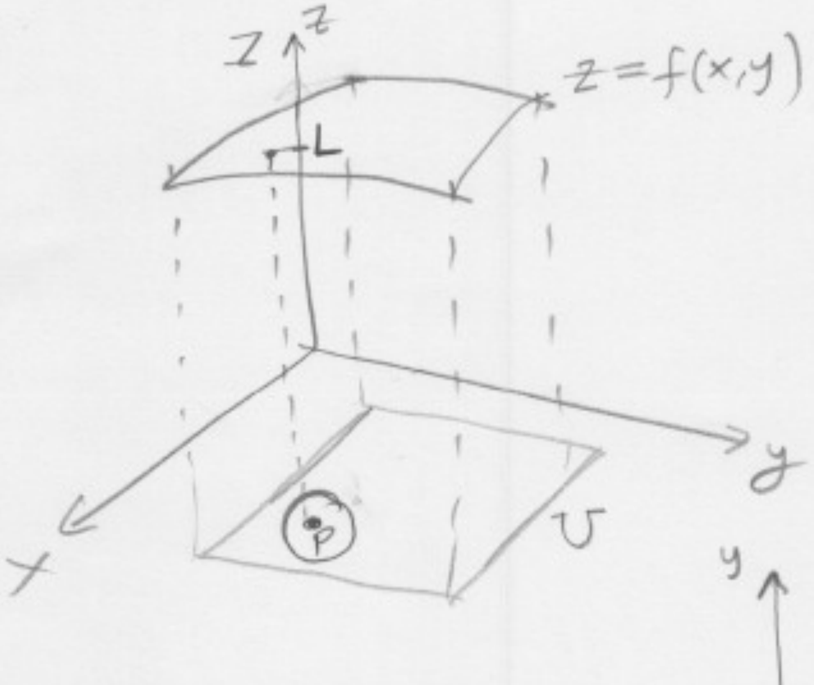
$$\lim_{x \to 0} \frac{e^x - 1 - \sin x}{x^2} = \lim_{x \to 0} \frac{e^x - \cos x}{2x}$$

$$= \lim_{x \to 0} \frac{e^x + \sin x}{2} = \frac{1}{2}.$$

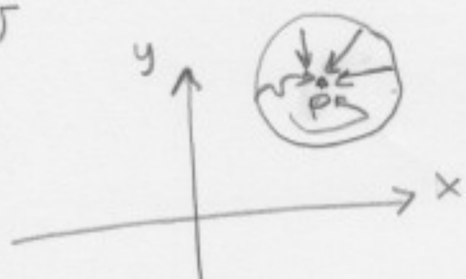
If we have $f: U \subset \mathbb{R}^2 \to \mathbb{R}$, then:

$$\lim_{(x,y) \to (x_0,y_0)} f(x,y) = L$$

means roughly speaking that "as the point (x,y) gets closer and closer to (x_0,y_0) then the value of $f(x,y)$ is closer and closer to L".

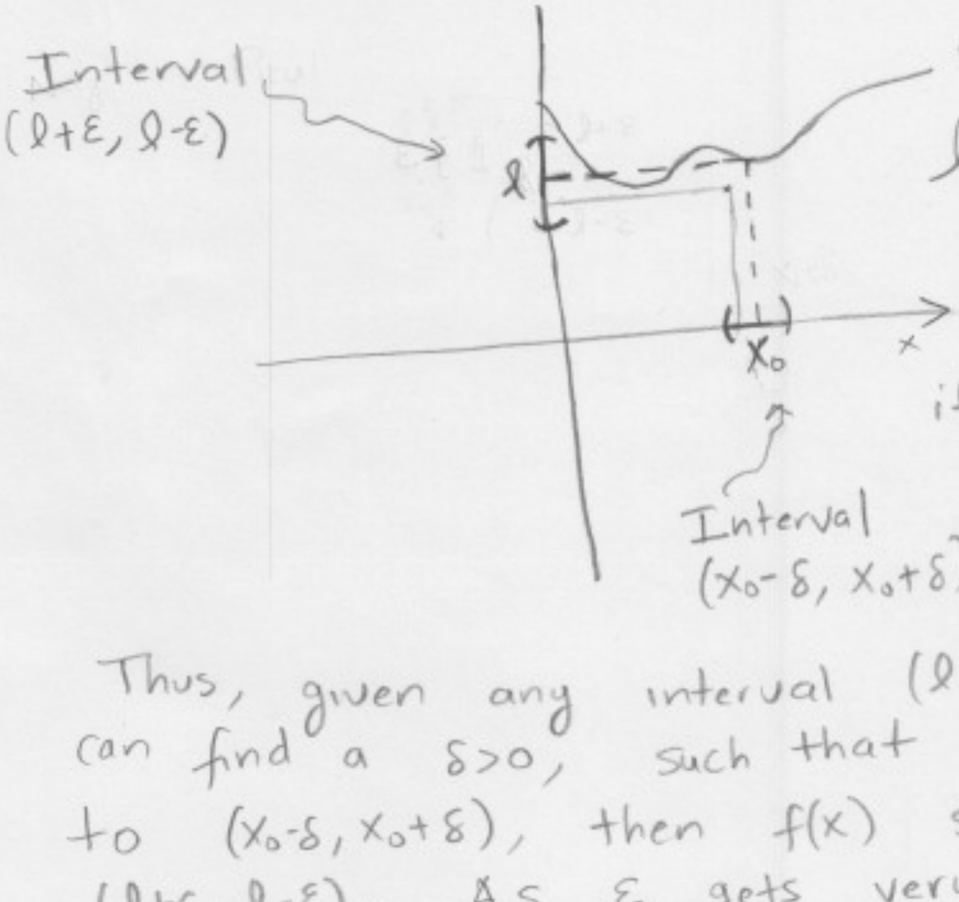


$P = (x_0, y_0)$



There is an infinite number of paths to approach P, not just left and right, as in functions of one variable. For this reason we need a mathematical rigorous way to define "closer and closer". This is accomplished with the following fundamental concept in mathematics:

Def. $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ means that:
given any $\varepsilon>0$, there exists $\delta>0$, such that:
if $\|(x,y)-(x_0,y_0)\|<\delta$ then $|f(x,y)-L|<\varepsilon$.



Picture in 1-dim

$\lim_{x\to x_0} f(x) = \ell \iff$

$\forall\varepsilon>0,\ \exists\delta>0$ such that

if $|x-x_0|<\delta \Rightarrow |f(x)-\ell|<\varepsilon$

Thus, given any interval $(\ell+\varepsilon, \ell-\varepsilon)$, we can find a $\delta>0$, such that if x belongs to $(x_0-\delta, x_0+\delta)$, then $f(x)$ stays inside $(\ell+\varepsilon, \ell-\varepsilon)$. As ε gets very small, the corresponding δ is also getting smaller, and in this way we define mathematically the "closer and closer" idea of limit.

To compute a limit in practice, many times we just apply the following Theorem, which can be proven using the rigorous definition of limit:

<u>Theorem</u>: If $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L_1$ & $\lim_{(x,y)\to(x_0,y_0)} g(x,y) = L_2$

then

(i) $\lim_{(x,y)\to(x_0,y_0)} (f+g) = L_1 + L_2$

(ii) $\lim_{(x,y)\to(x_0,y_0)} fg = L_1 L_2$

(iii) $\lim_{(x,y)\to(x_0,y_0)} \frac{f}{g} = \frac{L_1}{L_2}$, if $L_2 \neq 0$

(iv) Composition: if $h: \mathbb{R} \to \mathbb{R}$, $\lim_{t\to L_1} h(t) = \ell$ then

$$\lim_{(x,y)\to(x_0,y_0)} h \circ f = \ell$$

$$\mathbb{R}^2 \xrightarrow{f} \mathbb{R} \xrightarrow{h} \mathbb{R}$$

<u>Ex</u>: Compute $\lim_{(x,y)\to(0,0)} \frac{\sin xy}{x+1}$

From (ii) $\lim_{(x,y)\to(0,0)} xy = \left(\lim_{(x,y)\to(0,0)} x\right)\left(\lim_{(x,y)\to(0,0)} y\right) = 0 \cdot 0 = 0$

From (i) $\lim_{(x,y)\to(0,0)} x+1 = \lim_{(x,y)\to(0,0)} x + \lim_{(x,y)\to(0,0)} 1 = 0+1 = 1$

From (iv) $\sin xy \to 0$ as $(x,y)\to 0$. From (iii), $\frac{\sin xy}{x+1} \to \frac{0}{1} = 0$

<u>Ex</u>: Use the ε-δ definition to show that:

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{x^2+y^2}} = 0$$

We let $\varepsilon > 0$ be any positive number. We need to find a δ that satisfies the definition of limit. We notice the following:

$$\left| f(x,y) - 0 \right| = \left| \frac{x^2}{\sqrt{x^2+y^2}} - 0 \right|$$

$$= \frac{x^2}{\sqrt{x^2+y^2}}$$

$$= \frac{x^2\sqrt{x^2+y^2}}{x^2+y^2}$$; by multiplying the fraction by $\sqrt{x^2+y^2}$ in both numerator and denominator

$$\leq \sqrt{x^2+y^2}$$; since $\frac{x^2}{x^2+y^2} \leq 1$,

this fraction can not be greater than 1

$$= \|(x,y) - (0,0)\|$$

We have obtained:

$$\boxed{|f(x,y) - 0| \leq \|(x,y) - (0,0)\| \quad \to \quad (1)}$$

Therefore, from (1) it follows that if we choose $\delta = \frac{\varepsilon}{2}$ (or $\delta = \frac{\varepsilon}{4}$; choice of δ is not unique), then:

if $\|(x,y) - (0,0)\| < \delta$, from (1):

$$|f(x,y) - \underset{\substack{\uparrow \\ \text{limit}}}{0}| \leq \|(x,y) - (0,0)\| < \delta = \frac{\varepsilon}{2} < \varepsilon, \text{ and}$$

the definition is satisfied. This shows that the limit is 0 ■

Since applying the ε-δ definition to compute limits can be hard, for this class we will mostly restrict our computations to the following types of problems:

<u>Ex 1</u>: Show that $\lim\limits_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{x^2+y^2}} = 0$ using trick of polar coordinates.

$P = (0,0)$

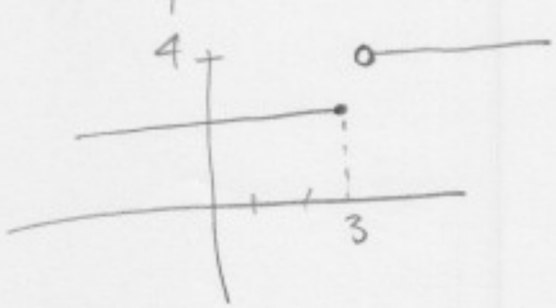
We write $x = r\cos\theta$, $y = r\sin\theta$. As (x,y) gets closer to $(0,0)$, we note that $r \to 0$ (the angle θ is rotating). Hence:

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \frac{r^2\cos^2\theta}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}} = \lim_{r\to 0} r\cos^2\theta = 0,$$

where we have used the squeze lemma:

$$\begin{array}{ccccc} 0 & \le & |r\cos^2\theta| & \le & r \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0 \end{array}$$

If the limit does not exist, it is enough to give two different paths where there is a jump. The picture in 1-d is the following:



lim from left is 3
lim from righ is 4
$3 \neq 4$
there is a jump so limit does not exist.

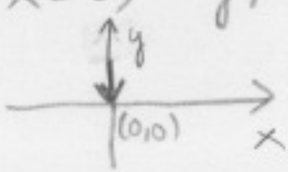
Examples in higher dimension:

<u>Ex 2</u>: Does the following limit exist?

$$\lim_{(x,y)\to(0,0)} \frac{\sin xy}{x^2+y^2}$$

This limit does not exist, since the behaviour changes upon different approaches to $(0,0)$.

(1) if we approach $(0,0)$ along the path $x=0$, $y\neq 0$ we have



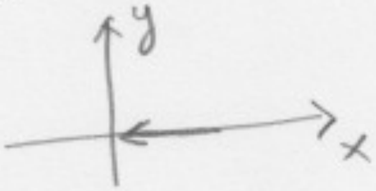
$$\lim_{\substack{x=0\\ y\to 0}} \frac{\sin xy}{x^2+y^2} = \lim_{\substack{y\to 0\\ x=0}} 0 = 0.$$

So the value of the function $f(x,y)$ at any point of the form $(0,\varepsilon)$ is

$$f(0,\varepsilon) = 0,$$

that is, $f(x,y)$ is identically zero along that path.

(2) If we approach $(0,0)$ along the path $y=0$, $x \neq 0$, we have:

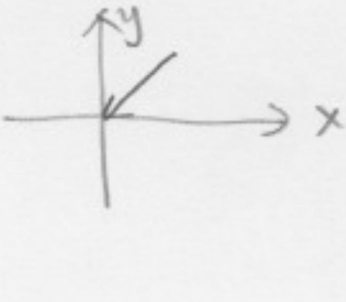


$$\lim_{\substack{y=0 \\ x\to 0}} \frac{\sin xy}{x^2+y^2} = \lim_{\substack{y=0 \\ x\to 0}} 0 = 0$$

that is, $f(x,y)$ is also identically zero along this path.

<u>REMARK</u>: From (1) and (2) we <u>CAN NOT</u> conclude that the limit is zero, because there is an infinite number of ways to approach $(0,0)$. Just because two of them gives 0, does not guarantee that other paths will also give zero. Indeed:

(3) if we approach $(0,0)$ along the path $y=x$ we have:



$$\lim_{\substack{y=x \\ x\to 0}} \frac{\sin xy}{x^2+y^2} = \lim_{x\to 0} \frac{\sin x^2}{2x^2} = \frac{1}{2}\lim_{x\to 0} \frac{\sin x^2}{x^2}$$

$$= \frac{1}{2}\lim_{\theta\to 0} \frac{\sin\theta}{\theta}, \text{ letting } \theta = x^2 \text{ as } x\to 0;\ \theta\to 0$$

$$= \frac{1}{2}\lim_{\theta\to 0} \cos\theta = \frac{1}{2}$$

Hence the function $f(x,y) = \frac{\sin xy}{x^2+y^2}$ approaches $\frac{1}{2}$ along the path $x=y$. Therefore, there is a jump from 0 to $\frac{1}{2}$ along these paths. We conclude that the limit does not exist.

Ex: Does the limit $\lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{x^2+2y^2}$ exist?

The limit does not exist since different approaches yield different limits.

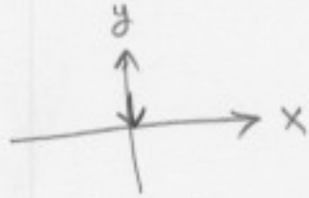
Note that the function is always 0 on the line $y=x$.

The function is always 4/3 on the line $y=-x$, $x\neq 0$, since $f(x,-x) = \frac{(2x)^2}{x^2+2x^2} = \frac{4x^2}{3x^2} = \frac{4}{3}$.

Ex: Investigate whether the following limit exists:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} \ ?$$

(1) We try along $x=0$, $y\to 0$



$$\lim_{\substack{y\to 0\\ y\neq 0\\ x=0}} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{\substack{x=0\\ y\to 0}} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{y\to 0} 0 = 0$$

(2) We try along $y=x$, $x\to 0$

$$\lim_{\substack{y=x\\ x\to 0}} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{x\to 0} \frac{x^2}{\sqrt{2x^2}} = \frac{1}{\sqrt{2}} \lim_{x\to 0} x = 0.$$

From (1) and (2) we can not conclude that the limit is zero. We can try other paths like $y=x^2$, or $y=0$, $x\to 0$, and we get again 0. Therefore, since we are having problems finding a jump, we suspect that the limit exist. To show that the limit exists we need to use the ε–δ definition or another method, like polar coordinates:

Let $x = r\cos\theta$, $y = r\sin\theta$,

$$\lim_{r\to 0} \frac{(r\cos\theta)(r\sin\theta)}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}} = \lim_{r\to 0} \frac{r^2 \sin\theta\cos\theta}{r}$$

$$= \lim_{r\to 0} r\cos\theta\sin\theta$$

$$= 0 \ ; \quad \text{since } 0 \le |r\sin\theta\cos\theta| \le r$$

Hence, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0.$

The use of polar coordinates accounts for "all possible paths" approaching $(0,0)$.

Ex : Does the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$ exist.

* We try along path $y = x$

$$\lim_{\substack{y=x\\ x\to 0}} \frac{x^2y}{x^4+y^2} = \lim_{x\to 0} \frac{x^2\cdot x}{x^4+x^2} = \lim_{x\to 0} \frac{x}{x^2+1} = 0.$$

* We try along path $y = x^2$

$$\lim_{\substack{y=x^2\\ x\to 0}} \frac{x^2y}{x^4+y^2} = \lim_{x\to 0} \frac{x^2\cdot x^2}{x^4+x^4} = \lim_{x\to 0} \frac{\cancel{x^4}}{2\cancel{x^4}} = \frac{1}{2}$$

There is a jump from 0 to $\frac{1}{2}$ so limit does not exist.

<u>Def</u>: The function $f(x,y)$ is continuous at $(x_0,y_0) \subset U$ if

(a) $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exists.

(b) $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

Analogous theorem (i)–(iv) holds for continuous functions.

<u>Ex</u>: We showed that $f(x,y) = \frac{xy}{\sqrt{x^2+y^2}}$ satisfies $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$. Thus if we re-define $f(x,y)$ as follows:

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0), \end{cases}$$

then $f(x,y)$ is also continuous at 0.

The function $f(x,y) = \frac{x^2y}{x^4+y^2}$ can not be defined at $(0,0)$ in any way to make it continuous.

<u>Ex</u>: $f(x,y) = \frac{1}{y}\left(e^{xy} - 1\right) = \frac{e^{xy}-1}{y}$

$\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exists as long as $y_0 \neq 0$

and the limit is $\frac{1}{y_0}\left(e^{x_0 y_0} - 1\right)$

Given a point $(x_0, 0)$, $x_0 \in \mathbb{R}$,

we compute:

$$\lim_{(x,y)\to(x_0,0)} f(x,y) = \lim_{(x,y)\to(x_0,0)} x\left[\frac{e^{xy}-1}{xy}\right]$$

$$= \left[\lim_{(x,y)\to(x_0,0)} x\right]\left[\lim_{(x,y)\to(x_0,0)} \frac{e^{xy}-1}{xy}\right]$$

$$= x_0 \cdot \lim_{\theta\to 0} \frac{e^{\theta}-1}{\theta},$$ since $(x,y)\to(x_0,0)$ and hence $xy \mapsto x_0 \cdot 0 = 0$

$$= x_0 \lim_{\theta\to 0} \frac{e^{\theta}}{1}$$

$$= x_0 \cdot 1 = x_0$$

Hence $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ always exist for any $(x_0,y_0)\in\mathbb{R}^2$. We can re-define:

$$f(x,y) = \begin{cases} \dfrac{e^{xy}-1}{y}, & (x,y) \text{ with } y \neq 0 \\ x, & \text{for any point } (x,0). \end{cases}$$

Hence with this definition, $f(x,y)$ is continuous at any point $(x,y) \in \mathbb{R}^2$.