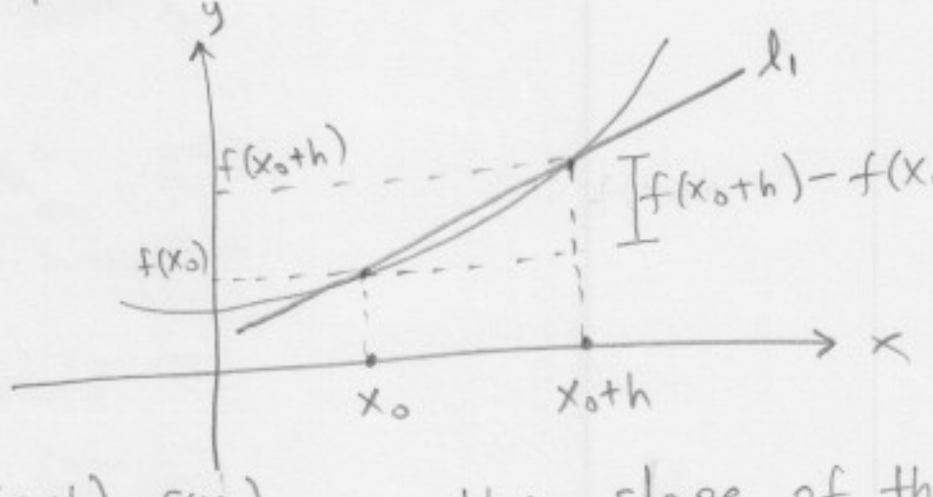
Section 2.3
Differentiation

Recall the concept of differentiation for functions of one variable $f: \mathbb{R} \to \mathbb{R}$

f is differentiable at x_0 if the following limit exists:

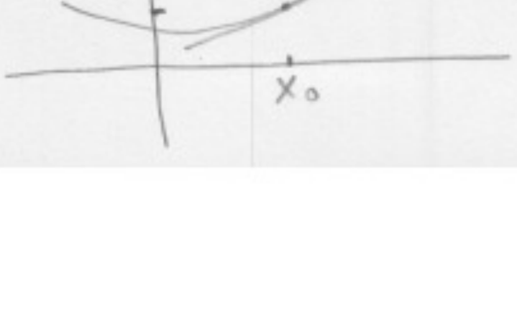
$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}.$$

This limit is denoted as $f'(x_0)$ and it is the slope of the tangent line to the graph at the point $(x_0, f(x_0))$. Indeed:



$\frac{f(x_0+h) - f(x_0)}{h}$ is the slope of the line l_1.

As h gets smaller, that slope is converging to the slope of the tangent line at $(x_0, f(x_0))$



Partial derivatives:
Definition and geometrical meaning.

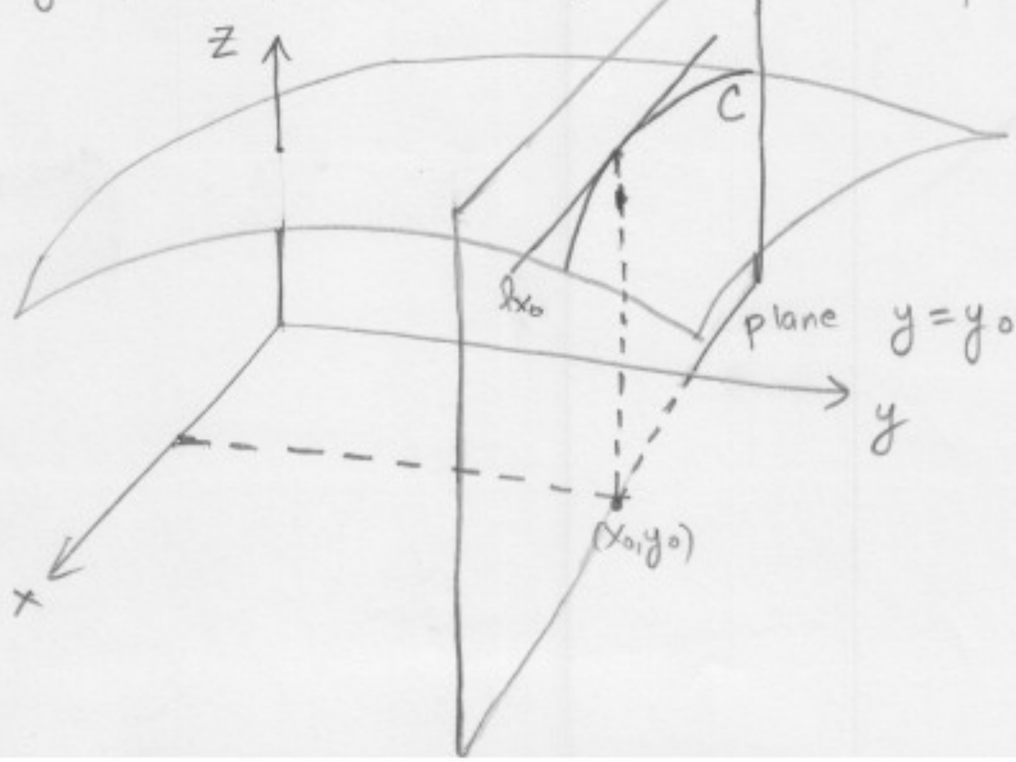
Let $f: U \subset \mathbb{R}^2 \to \mathbb{R}$, where U is an open set in $\mathbb{R}^2$. Let $(x_0, y_0) \in U$. We define:

$$\frac{\partial f}{\partial x}(x_0, y_0) := \lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}.$$

If this limit exists, it is called the partial derivative of f, with respect to x, at (x_0, y_0); and it is denoted with the symbol:

$$\frac{\partial f}{\partial x}(x_0, y_0) \quad \text{or} \quad f_x(x_0, y_0).$$

This number is the slope of the tangent line ℓ_{x_0} to the curve C that is the intersection of the graph of $z = f(x, y)$ and the plane $y = y_0$.



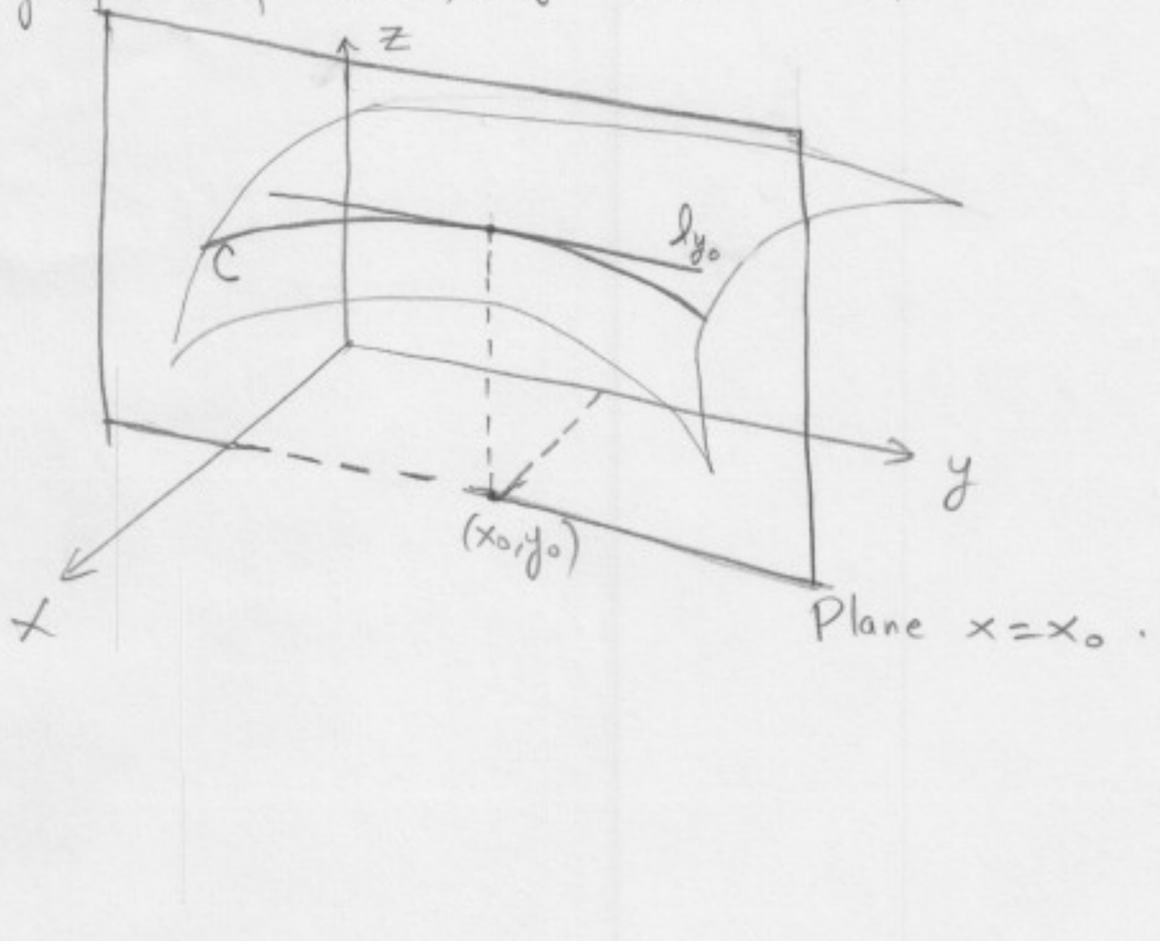
Let $f : U \subset \mathbb{R}^2 \to \mathbb{R}$, where U is an open set in $\mathbb{R}^2$. Let $(x_0, y_0) \in U$. We define:

$$\frac{\partial f}{\partial y}(x_0,y_0) := \lim_{h \to 0} \frac{f(x_0, y_0+h) - f(x_0,y_0)}{h}$$

If this limit exists, it is called the partial derivative of f, with respect to y, at (x_0,y_0); and it is denoted with the symbol:

$$\frac{\partial f}{\partial y}(x_0,y_0) \quad \text{or} \quad f_y(x_0,y_0).$$

This number is the slope of the tangent line l_{y_0} to the curve C that is the intersection of the graph of $z=f(x,y)$ and the plane $x=x_0$.



For practical computations, many times we can use theorems as in Calculus of 1 variable in order to compute the partial derivatives. This is possible since the partial derivative is defined by fixing a variable (i.e. a plane), and taking derivatives, as in Calculus of 1 variable, in a plane.

<u>Ex</u>: Let $f(x,y) = e^{xy^2} + (x^2+3y^3)^{10}$

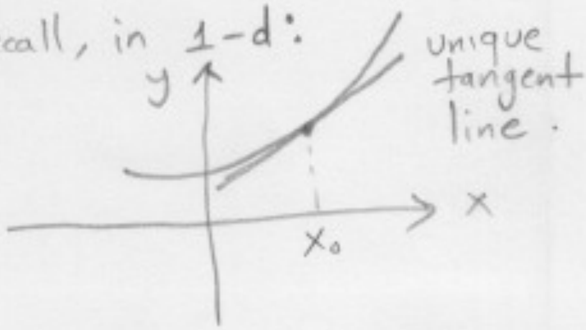
$$\frac{\partial f}{\partial x} = y^2 e^{xy^2} + 10\,(x^2+3y^3)^9 (2x)$$

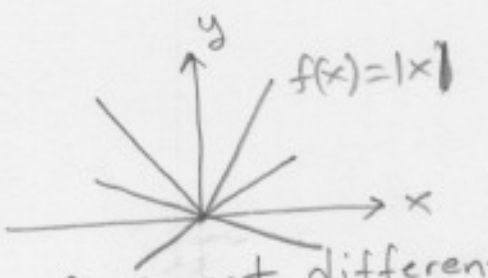
$$\frac{\partial f}{\partial y} = 2xy\, e^{xy^2} + 10\,(x^2+3y^3)^9 \cdot 9y^2$$

Differentiability of $f: U \subset \mathbb{R}^2 \to \mathbb{R}$ at $(x_0,y_0) \in U$

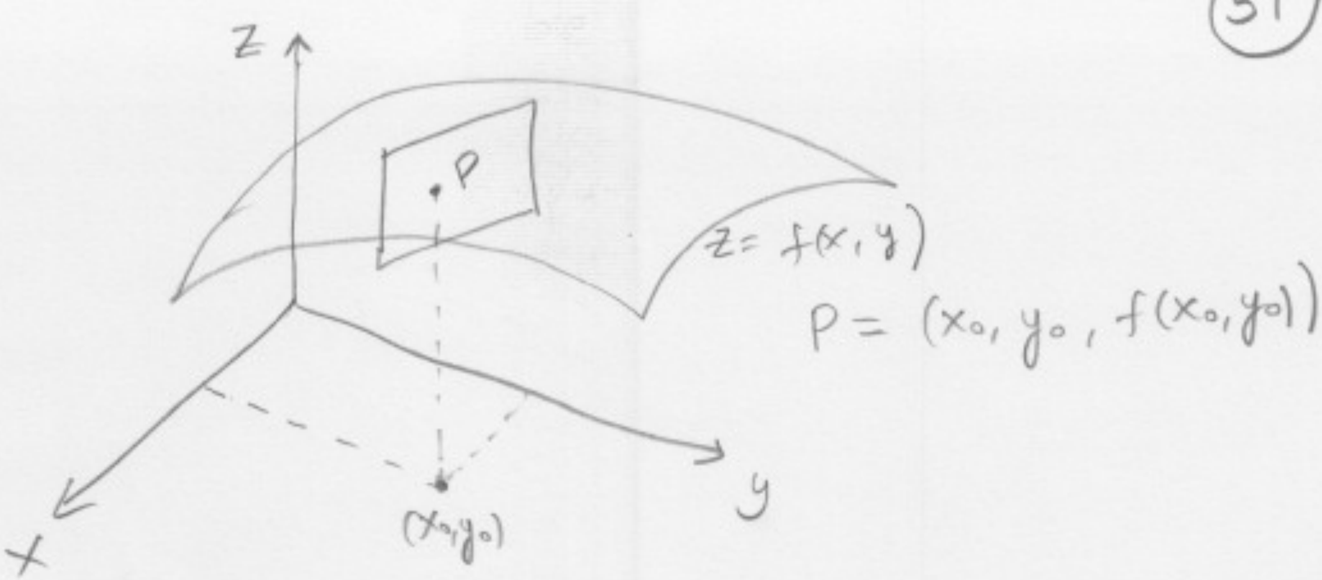
A function $f: U \subset \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $(x_0,y_0) \in U$ if there exists a unique tangent plane at the point $(x_0, f(x_0))$.

Recall, in 1-d:

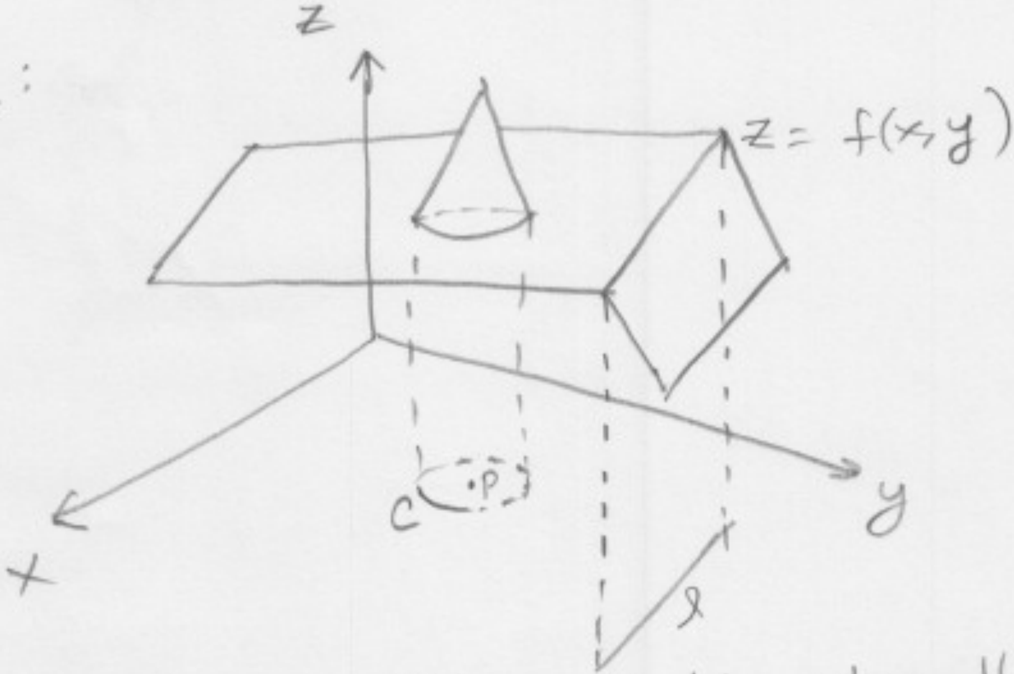




f is not differentiable at 0. Note that we can not find a unique tangent line at $(0,0)$.

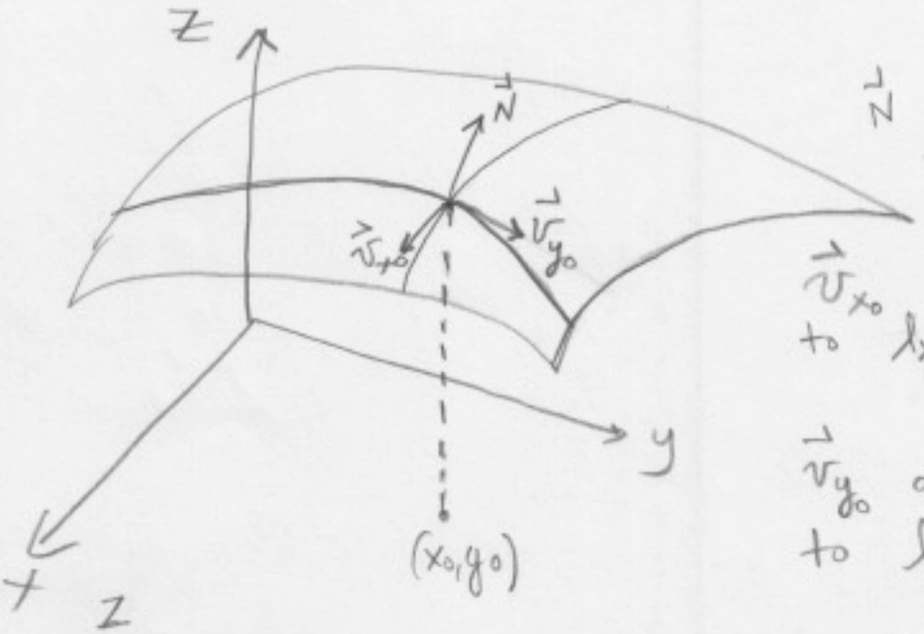


Ex:



f is not differentiable along the line ℓ.
f is not differentiable along the circle C or at the point P.

If f is differentiable at (x_0, y_0); that is, there exists a unique tangent plane at $(x_0, y_0, f(x_0, y_0))$, and we want to compute the equation of this plane, we proceed as follows:

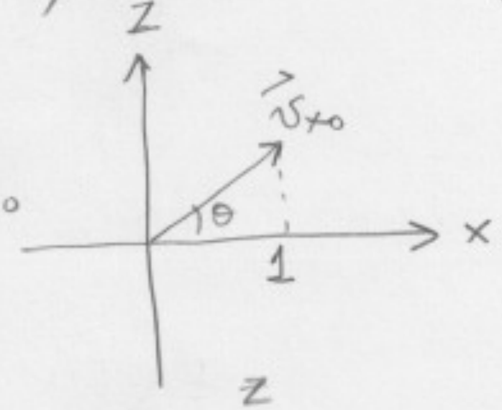


$\vec{N}$ perpendicular to tangent plane

$\vec{v}_{x_0}$ a vector parallel to l_{x_0} (see Page 48)

$\vec{v}_{y_0}$ a vector parallel to l_{y_0} (see page 49)

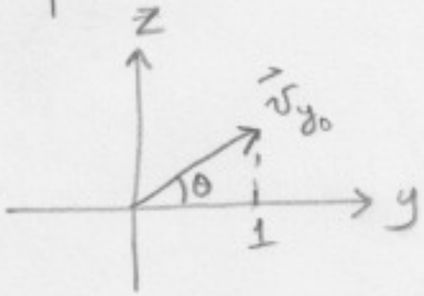
Fix $y = y_0$



$$\tan\theta = \frac{\partial f}{\partial x}(x_0, y_0)$$

$$\Rightarrow \vec{v}_{x_0} = \left(1, 0, \frac{\partial f}{\partial x}(x_0, y_0)\right)$$

Fix $x = x_0$



$$\tan\theta = \frac{\partial f}{\partial y}(x_0, y_0)$$

$$\vec{v}_{y_0} = \left(0, 1, \frac{\partial f}{\partial y}(x_0, y_0)\right)$$

$$\vec{N} = \vec{v}_{x_0} \times \vec{v}_{y_0} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{\partial f}{\partial x}(x_0, y_0) \\ 0 & 1 & \frac{\partial f}{\partial y}(x_0, y_0) \end{vmatrix}$$

$$= \vec{i}\left(-\frac{\partial f}{\partial x}(x_0, y_0)\right) - \vec{j}\left(\frac{\partial f}{\partial y}(x_0, y_0)\right) + \vec{k}$$

$$= \left(-\frac{\partial f}{\partial x}(x_0, y_0),\ -\frac{\partial f}{\partial y}(x_0, y_0),\ 1\right)$$

Thus, the equation of the tangent plane is (letting $z_0 = f(x_0, y_0)$):

$$-\frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) - \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) + (z - z_0) = 0$$

or:

$$\boxed{z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)}$$

Ex: Let S be the surface which is the graph of $z = e^{xy^2}\ln(x^2+y^2+1)$. Find the equation of the tangent plane when $(x, y) = (0, 1)$

$$f(0,1) = \ln 2$$

$$\frac{\partial f}{\partial x} = y^2 e^{xy^2}\ln(x^2+y^2+1) + e^{xy^2} \cdot \frac{2x}{x^2+y^2+1}$$

$$\frac{\partial f}{\partial x}(0,1) = \ln 2$$

$$\frac{\partial f}{\partial y} = 2xy\, e^{xy^2}\ln(x^2+y^2+1) + \frac{2y}{x^2+y^2+1} e^{xy^2}$$

$$\frac{\partial f}{\partial y}(0,1) = 1$$

$\therefore$ $\boxed{z = \ln 2 + (\ln 2)x + y - 1}$, is the equation of the plane.

Ex: Let $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Compute all partial derivatives.

If $(x,y) \neq (0,0)$ we compute the partial derivative at (x,y) as follows:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[xy\,(x^2+y^2)^{-1/2} \right]$$
$$= y\,(x^2+y^2)^{-1/2} - \frac{1}{2}xy\,(x^2+y^2)^{-3/2}(2x)$$
$$= \frac{y}{\sqrt{x^2+y^2}} - \frac{x^2y}{(x^2+y^2)^{3/2}}.$$

If $(x,y) = (1,1)$, for example,

$$\frac{\partial f}{\partial x}(1,1) = \frac{1}{\sqrt{2}} - \frac{1}{2^{3/2}}$$

If $(x,y) = (0,0)$ we have to use the original definition as a limit:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

Similarly:

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} =$$
$$= \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0.$$

Therefore, we have shown that $f(x,y)$ has the following properties

1.- $f(x,y)$ is continuous at every point $(x_0,y_0) \in \mathbb{R}^2$, including the point $(0,0)$.

2.- $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ both exist at every point $(x_0,y_0) \in \mathbb{R}^2$, including the point $(0,0)$

3.- However, we will see next that f is not differentiable at $(0,0)$; that is, there is NOT a unique tangent plane that approximates the graph at $(0,0)$.

We will also see that f is differentiable at every other point $(x,y) \in \mathbb{R}^2$, $(x,y) \neq (0,0)$.