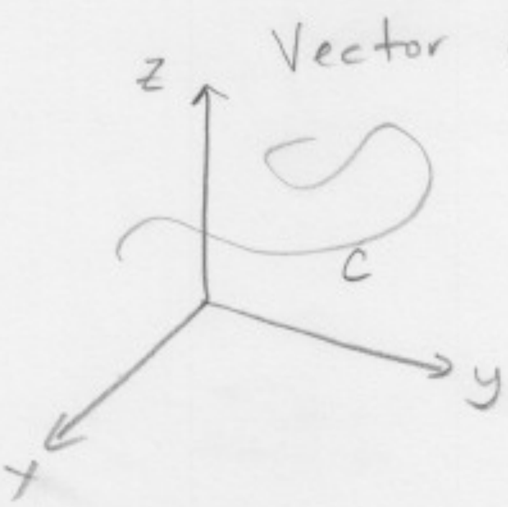
Section 4.1
Vector functions.




$\vec{r}(t) = (x_1(t), \dots, x_n(t))$ Position vector

$\vec{r}\,'(t) = (x_1'(t), \dots, x_n'(t))$ Velocity Vector

$\vec{r}\,''(t) = (x_1''(t), \dots, x_n''(t))$ Acceleration Vector

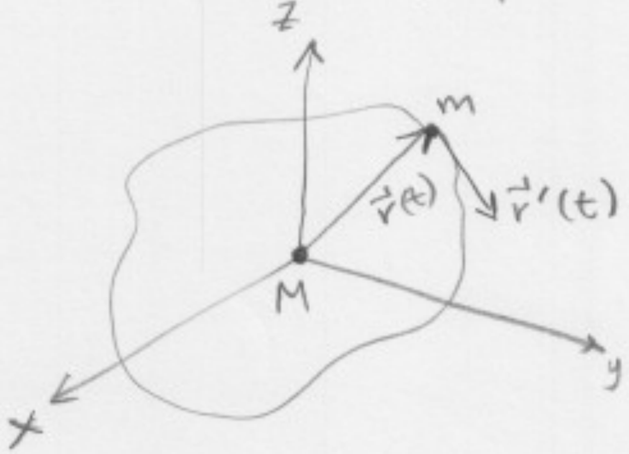
If a particle is traveling along the path C with constant speed (that is, $\|\vec{r}\,'(t)\| = c$, $c \neq 0$, for every $t$) we have that $\vec{r}\,''(t)$ is orthogonal to $\vec{r}\,'(t)$.
In order to see this we compute:

$$\|\vec{r}\,'(t)\| = c$$
$$\Rightarrow \|\vec{r}\,'(t)\|^2 = c^2$$
$$\Rightarrow \vec{r}\,'(t) \cdot \vec{r}\,'(t) = c^2$$
$$\frac{d}{dt}\left(\vec{r}\,'(t) \cdot \vec{r}\,'(t)\right) = 0$$
$$\vec{r}\,'(t) \cdot \vec{r}\,''(t) + \vec{r}\,'(t) \cdot \vec{r}\,''(t) = 0$$
$$2\,\vec{r}\,'(t) \cdot \vec{r}\,''(t) = 0$$

Hence, $\vec{r}\,''(t) \perp \vec{r}\,'(t)$, at each point where $\vec{r}\,''(t) \neq \vec{0}$.

Suppose we have a mass $M$ at the origin. Then the force on a mass $m$ at a point $\vec{r}(t)$ is given by:

$$-\frac{GmM\,\vec{r}(t)}{\|\vec{r}(t)\|^3}$$




Newton's law says that $\vec{F} = m\vec{a}$, that is,

$$m\vec{r}\,''(t) = -\frac{GmM\,\vec{r}}{\|\vec{r}\|^3}$$

Suppose a mass $m$ is moving with constant speed $\|\vec{r}\,'(t)\| = c$ in a counterclockwise circular path of radius $r_0$ in the $xy$-plane. Then:

$$\vec{r}(t) = \left(r_0 \cos \frac{c}{r_0}(t-t_0),\ r_0 \sin \frac{c}{r_0}(t-t_0)\right),$$

where $t_0$ is such that $\vec{r}(t_0) = (r_0, 0)$. The factor $\frac{c}{r_0}$ is to make $\|\vec{r}\,'(t)\| = c$

Indeed:

$$\vec{r}'(t) = \left(-r_0 \frac{c}{r_0} \sin \frac{c}{r_0}(t-t_0),\ r_0 \frac{c}{r_0} \cos \frac{c}{r_0}(t-t_0)\right)$$

and

$$\|\vec{r}'(t)\| = \sqrt{c^2 \sin^2 \frac{c}{r_0}(t-t_0) + c^2 \cos^2 \frac{c}{r_0}(t-t_0)} = c$$

The acceleration is:

$$\vec{r}''(t) = \left(-\frac{c^2}{r_0} \cos \frac{c}{r_0}(t-t_0),\ \frac{-c^2}{r_0} \sin \frac{c}{r_0}(t-t_0)\right)$$

$$= -\frac{c^2}{r_0^2}\vec{r}(t)$$

In particular, note that $\vec{r}''(t)$ is perpendicular to $\vec{r}'(t)$.

The quantity $-\frac{mc^2}{r_0^2}\vec{r}(t)$ is the centripetal force.

## Arc lenght

The quantity $\ell = \int_{t_0}^{t_1} \|\vec{r}'(t)\| \, dt$ is the arc lenght of the curve $C$ given by $\vec{r}(t)$, $t_0 \leq t \leq t_1$.