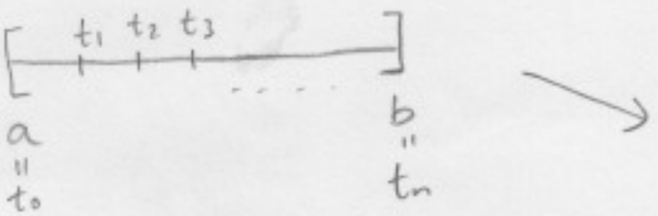
Section 4.2

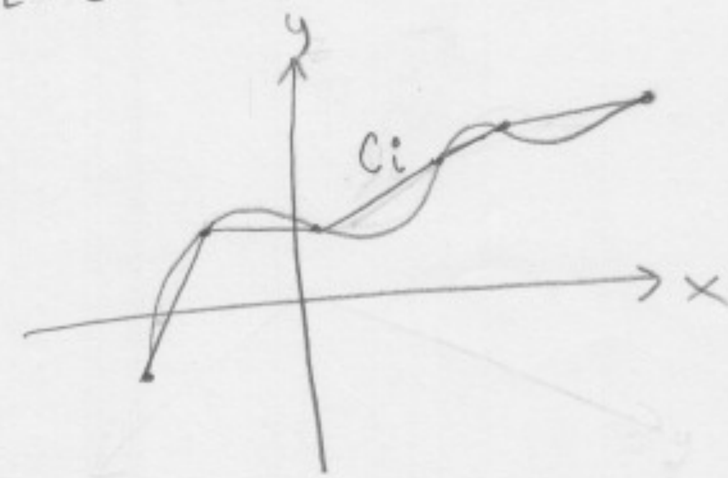
Arc lenght

The quantity $\ell = \int_{t_0}^{t_1} \|\vec{r}\,'(t)\|\, dt$ is the arc lenght of the curve C given by $\vec{r}(t)$, $t_0 \le t \le t_1$

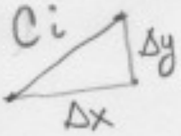
In order to see that this formula is true we partition the curve in to pieces of line:

$$\vec{r}(t) = (x(t), y(t)) : [a,b] \to \mathbb{R}$$





$$\Delta t = t_{i+1} - t_i = \frac{b-a}{n}$$



$$\begin{aligned} l_i &= \sqrt{\Delta x^2 + \Delta y^2} \\ &= \sqrt{(x(t_{i+1}) - x(t_i))^2 + (y(t_{i+1}) - y(t_i))^2} \end{aligned}$$

$$\ell \cong \sum_{i=0}^{n-1} l_i = \sum_{i=0}^{n-1} \sqrt{(x(t_{i+1}) - x(t_i))^2 + (y(t_{i+1}) - y(t_i))^2}$$

$\Rightarrow$

$$\ell \cong \sum_{i=0}^{n-1} \sqrt{\left[\frac{(x(t_{i+1})-x(t_i))}{\Delta t}\right]^2 + \left[\frac{(y(t_{i+1})-y(t_i))}{\Delta t}\right]^2} \ \Delta t$$

We let $n \to \infty$ (or $\Delta t \to 0$) to get the exact lenght

$$\ell = \lim_{\Delta t \to 0} \sum_{i=0}^{n-1} \ell_i = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

$$= \int_a^b \|\vec{r}\,'(t)\|\, dt$$

(Note that $x(t_{i+1}) = x(t_i + \Delta t)$, $y(t_{i+1}) = y(t_i + \Delta t)$).

<u>Ex</u>: $\vec{r}(t) = (\cos t, \sin t, t^2)$, $0 \le t \le \pi$

$$\vec{r}\,'(t) = (-\sin t, \cos t, 2t)$$

$$\|\vec{r}\,'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 4t^2}$$

$$= 2\sqrt{t^2 + \tfrac{1}{4}}$$

$$\ell = 2\int_0^{\pi} \sqrt{t^2 + \tfrac{1}{4}}\, dt.$$

This can be evaluated by making the substitution $t = \frac{1}{4}\tan x$, $dt = \frac{1}{4}\sec^2 x\, dx$.

Parametrization with respect to arc length.

Problem: Given C, find a parametrization $\vec{r}(t)$ such that $\|\vec{r}\,'(t)\|=1$ (unit speed path). Notice that for a unit speed curve, $\vec{r}(t)$ $a \le t \le b$, then:

$$\ell = \int_a^b \|\vec{r}\,'(t)\|\,dt = \int_a^b dt = b-a.$$

Ex: Let $\vec{r}(t) = (1, 3t^2, t^3)$, $0 \le t \le 1$.
Reparametrize with respect to arc lenght.

$\vec{r}\,'(t) = (0, 6t, 3t^2)$. This is not a unit speed parametrization.

$$s(t) = \int_0^t \sqrt{36\tau^2 + 9\tau^4}\,d\tau = \frac{3}{2}\int_0^t 2\tau\sqrt{4+\tau^2}\,d\tau$$

$$= \frac{3}{2}\left[\frac{(4+\tau^2)^{3/2}}{3/2}\right]_0^t = (4+t^2)^{3/2} - 8$$

$\therefore$ $s(t) = (4+t^2)^{3/2} - 8$

We can solve $t(s)$; i.e, t in terms of s.

$$(4+t^2)^{3/2} = s+8$$
$$4+t^2 = (s+8)^{2/3}$$
$$t = \sqrt{(s+8)^{2/3}-4}$$

$$\Rightarrow \vec{R}(s) = \vec{r}(t(s))$$
$$= \left(1,\ 3(s+8)^{3/3}-12,\ \left((s+8)^{2/3}-4\right)^{3/2}\right).$$

• Hence, in order to reparametrize with respect to arc lenght we take the function $s(t)$:

$$s(t) = \int_0^t \|\vec{r}\,'(\tau)\|\, d\tau$$ (lengh of curve between $\vec{r}(0)$ and $\vec{r}(t)$).

$$\Rightarrow \frac{ds}{dt} = \|\vec{r}\,'(t)\|$$; using the fundamental theorem of Calculus

Since $s'(t) > 0$, $s(t)$ is always increasing and hence the inverse function $t(s)$ exists

We define:

$$\boxed{\vec{R}(s) = \vec{r}(t(s))}$$

When we parametrize with respect to arc lenght, <u>the speed is always 1</u>:

$$\frac{d\vec{R}}{ds} = \frac{d}{ds}\vec{r}(t(s))$$

$$= \frac{d\vec{r}}{dt} \cdot \frac{dt}{ds}; \quad \text{using chain rule}$$

$$= \frac{d\vec{r}/dt}{ds/dt}; \quad \text{since } \frac{dt}{ds}\cdot\frac{ds}{dt} = 1,$$

because, if $f = t(s)$, and $g = s(t)$,

$$= \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$$

$(f \circ g)(t) = t$

$\frac{d}{dt}(f \circ g)(t) = 1$

$f'(g(t))\, g'(t) = 1$

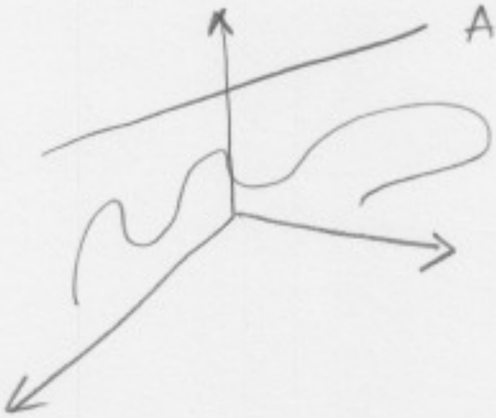
$\frac{dt}{ds}(s(t)) \cdot \frac{ds}{dt} = 1$

$\Rightarrow \frac{dt}{ds} = \frac{1}{ds/dt}$

Hence, $\|\vec{R}'(s)\| = \left\| \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} \right\| = 1$.

As explained in class, an application of the arc lenght parametrization is to study "curvature" of paths. Our textbook, in later chapters, also deals with "curvature" of surfaces.

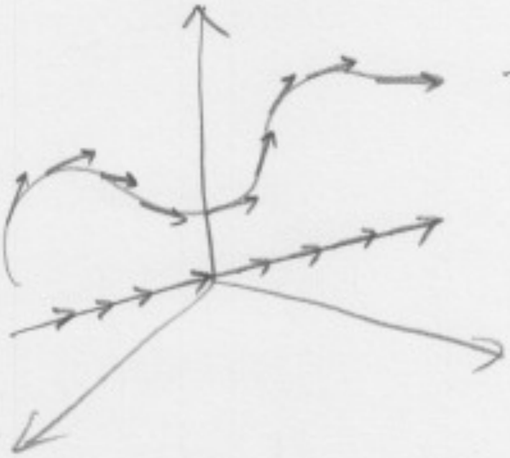
Curvature



A line has no curvature

This path has curvature, more curvature in some parts, less curvature on others.

Question: How do we measure curvature?



$\vec{r}(t) = (x(t), y(t), z(t))$

We define the tangent unit vector as:

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$$

$\vec{T}(t)$ is a unit vector in the same direction as the velocity vector. In a line, $\vec{T}(t)$ is always the same and hence $\frac{d}{dt}\vec{T}(t) = 0$, which says there is no curvature. Thus, we can measure curvature by looking at the derivative of $\vec{T}(t)$, which tell us how the tangent vector $\vec{T}(t)$ is changing. But, given a path C, we have learned that we can parametrize with different formulas $\vec{r}(t)$ because we can travel along C with different velocities. Indeed, if we travel with $\vec{r}(t) = (1+2t, 1+2t, 1+2t)$, we are traveling with velocity $\vec{r}\,'(t) = (2,2,2)$ and speed $\|\vec{r}\,'(t)\| = \sqrt{12}$. But we can travel the same line with $\vec{R}(s) = \left(1+\frac{s}{\sqrt{3}}, 1+\frac{s}{\sqrt{3}}, 1+\frac{s}{\sqrt{3}}\right)$, and hence with velocity $\vec{R}\,'(s) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and speed $\|\vec{R}\,'(s)\| = 1$. Hence $\vec{R}(s)$ is the parametrization with respect to arc length. Indeed, if we apply the procedure in page 142 to $\vec{r}(t) = (1+2t, 1+2t, 1+2t)$ we have $s(t) = \int_0^t \|\vec{r}\,'(\tau)\|\,d\tau = \int_0^t \sqrt{12}\,dt = \sqrt{12}\,t$. So, $s(t) = \sqrt{12}\,t$ and $t(s) = \frac{s}{\sqrt{12}}$. Hence:

$$\vec{R}(s) = \vec{r}(t(s)) = \vec{r}\left(\frac{s}{\sqrt{12}}\right) = \left(1+\frac{2s}{\sqrt{12}}, 1+\frac{2s}{\sqrt{12}}, 1+\frac{2s}{\sqrt{12}}\right)$$
$$= \left(1+\frac{s}{\sqrt{3}}, 1+\frac{s}{\sqrt{3}}, 1+\frac{s}{\sqrt{3}}\right).$$

Therefore, in order to avoid ambiguity, we can use <u>the unit speed path</u> to measure curvature. Since there is only one unit speed path, we can now define:

<u>Definition</u>: Given a path parametrized with respect to arc length, $\vec{r}(s)$, we define the function, $\kappa(s)$, the curvature at s, as follows:

$$\kappa(s) = \left\| \frac{d}{ds} \vec{T}(s) \right\|$$

$$\Rightarrow \quad \kappa(s) = \left\| \frac{d}{ds} \vec{r}'(s) \right\| ; \quad \text{since } \vec{T}(s) = \frac{\vec{r}'(s)}{\|\vec{r}'(s)\|} \text{ and}$$

$\|\vec{r}'(s)\| = 1$, since $\vec{r}(s)$ is the unit speed path

$$\Rightarrow \quad \boxed{\kappa(s) = \| \vec{r}''(s) \|}$$

<u>Ex.</u> If $\vec{r}(t)$ is given in terms of some parameter t and $\vec{r}'(t)$ is never $\vec{0}$, show that:

$$\kappa = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}.$$

<u>Solution</u>: We consider the tangent unit vector:

$$\vec{T}(s(t))$$

Using the chain rule:

$$\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds} \cdot \frac{ds}{dt} = \frac{d\vec{T}}{ds} \|\vec{r}'(t)\| ; \quad \text{since } s'(t) = \|\vec{r}'(t)\|$$

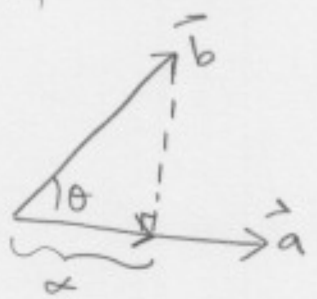
Therefore:

$$\|\vec{T}'(t)\| = \|\vec{r}'(t)\| \, \left\| \frac{d\vec{T}}{ds} \right\|$$

Hence:

$$\left\| \frac{d\vec{T}}{ds} \right\| = K = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|}$$

We need to compute $\vec{T}'(t)$. We first recall our formula to compute projections:

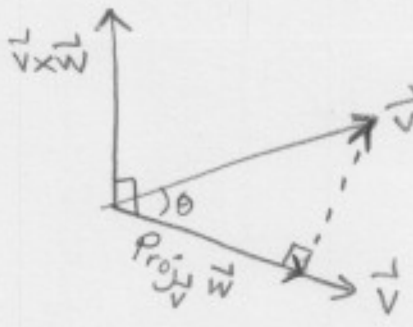


$$\cos\theta = \frac{\alpha}{\|\vec{b}\|}$$

$$\alpha = \cos\theta \, \|\vec{b}\| = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|} \|\vec{b}\| = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|}$$

$$\boxed{\mathrm{Proj}_{\vec{a}}\vec{b} = \alpha \frac{\vec{a}}{\|\vec{a}\|} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|}\frac{\vec{a}}{\|\vec{a}\|} = \left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|^2}\right)\vec{a}}, \quad (1)$$

Let $\vec{v}$ and $\vec{w}$ any two vectors, we can use (1) to obtain a formula for $\|\vec{v}\times\vec{w}\|$:



$$\|\vec{v}\times\vec{w}\| = \|\vec{v}\|\,\|\vec{w}\|\sin\theta \text{ ; from Chapter 1}$$

$$\mathrm{Proj}_{\vec{v}}\vec{w} + (\vec{w} - \mathrm{Proj}_{\vec{v}}\vec{w}) = \vec{w}$$

(2)

We compute:

$\|\vec{w}\|\sin\theta = \|\vec{w} - \text{Proj}_{\vec{v}}\vec{w}\|$; see right triangle in previous picture

$$= \left\|\vec{w} - \left(\frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|^2}\right)\vec{v}\right\|; \quad \text{From } (1)$$

From (2):

$$\|\vec{v}\times\vec{w}\| = \|\vec{v}\|\cdot\|\vec{w}\|\sin\theta$$
$$= \|\vec{v}\|\ \left\|\vec{w} - \left(\frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|^2}\right)\vec{v}\right\|$$

We have found:

$$\boxed{\|\vec{v}\times\vec{w}\| = \|\vec{v}\|\ \left\|\vec{w} - \left(\frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|^2}\right)\vec{v}\right\| \qquad (3)}$$

We will use the following:

Lemma: Let $\rho(t) = (x(t), y(t), z(t))$. Then:

$$\frac{d}{dt}\left(\frac{\rho(t)}{\|\rho(t)\|}\right) = \frac{\rho'(t)}{\|\rho(t)\|} - \frac{\rho(t)\cdot\rho'(t)}{\|\rho(t)\|^3}\rho(t)$$

$$\frac{d}{dt}\left(\frac{\rho(t)}{\|\rho(t)\|}\right) = \frac{d}{dt}\left(\frac{x(t)}{(x^2+y^2+z^2)^{1/2}}, \frac{y(t)}{(x^2+y^2+z^2)^{1/2}}, \frac{z(t)}{(x^2+y^2+z^2)^{1/2}}\right)$$

$$\frac{d}{dt}\left[x(t)\,(x^2+y^2+z^2)^{-1/2}\right] = x'(t)\,(x^2+y^2+z^2)^{-1/2} - \frac{1}{2}x(t)\,(x^2+y^2+z^2)^{-3/2}\cdot(2xx'+2yy'+2zz')$$

$$= \frac{x'(t)}{\|\rho(t)\|} - \frac{x(t)}{\|\rho(t)\|^3}\rho(t)\cdot\rho'(t)$$

Similarly:

$$\frac{d}{dt}\left[y(t)\,(x^2+y^2+z^2)^{-1/2}\right] = \frac{y'}{\|\rho(t)\|} - \frac{y(t)}{\|\rho(t)\|^3}\,\rho(t)\cdot\rho'(t)$$

and

$$\frac{d}{dt}\left[z(t)\,(x^2+y^2+z^2)^{-1/2}\right] = \frac{z'}{\|\rho(t)\|} - \frac{z(t)}{\|\rho(t)\|^3}\,\rho(t)\cdot\rho'(t)$$

Putting together the terms in the vector form we obtain:

$$\frac{d}{dt}\left(\frac{\rho(t)}{\|\rho(t)\|}\right) = \frac{\rho'(t)}{\|\rho(t)\|} - \rho(t)\cdot\rho'(t)\,\frac{\rho(t)}{\|\rho(t)\|^3}\,. \quad \blacksquare$$

We now apply the previous Lemma to $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$; that is, $\rho(t)$ will be $\vec{r}\,'(t)$:

$$\frac{d}{dt}\left(\frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}\right) = \frac{\vec{r}\,''(t)}{\|\vec{r}\,'(t)\|} - \frac{\vec{r}\,'(t)\cdot\vec{r}\,''(t)}{\|\vec{r}\,'(t)\|^3}\,\vec{r}\,'(t)$$

Hence

$$\|\vec{T}\,'(t)\| = \frac{1}{\|\vec{r}\,'(t)\|}\cdot\left\|\vec{r}\,''(t) - \frac{\vec{r}\,'(t)\cdot\vec{r}\,''(t)}{\|\vec{r}\,'(t)\|^2}\,\vec{r}\,'(t)\right\|$$

$$= \frac{1}{\|\vec{r}\,'(t)\|}\cdot\frac{\|\vec{r}\,'(t)\times\vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|}\,; \quad \text{using (3) with } \vec{v}=\vec{r}\,'(t),\ \vec{w}=\vec{r}\,''(t)$$

$$= \frac{\|\vec{r}\,'(t)\times\vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^2}$$

We now go back to finish our Ex. in page 146. We have, See page 147, that:

$$\kappa = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|}$$

$$= \frac{\dfrac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^2}}{\|\vec{r}'(t)\|}\text{; we have just computed } \|\vec{T}'(t)\|$$

$$= \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$$

Given any parametrization $\vec{r}(t)$, we have obtained a formula for the curvature in terms of t:

$$\kappa(t) = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3} \quad \blacksquare.$$