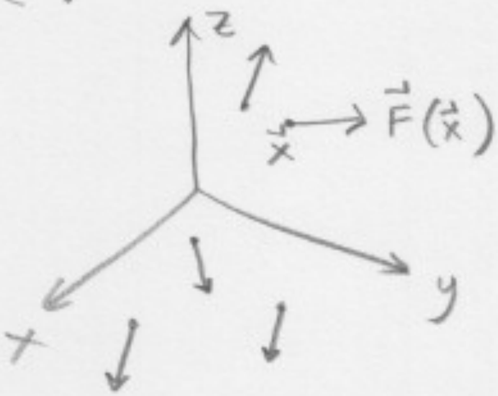
Section 4.3
Vector fields

A vector field is a function:

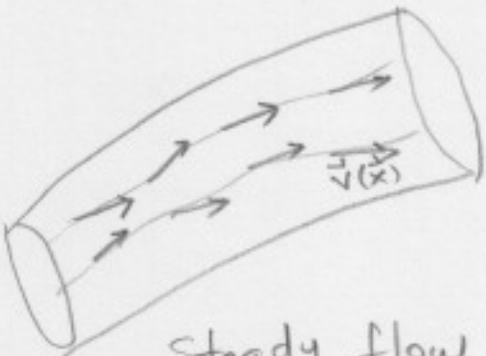
$$\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$$

$$\vec{F}(\vec{x}) = (F_1(\vec{x}), \ldots, F_n(\vec{x})), \quad \vec{x} \in \mathbb{R}^n$$

In $\mathbb{R}^3$:



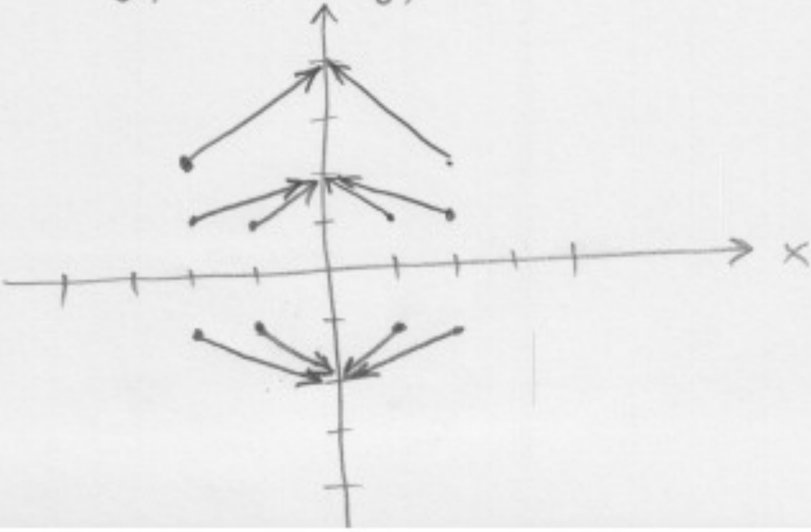
Ex:



Steady flow

$\vec{v}(\vec{x})$ is a vector field describing the velocity of flow in a pipe.

Ex: $\vec{F}(x, y) = (-x, y) : \mathbb{R}^2 \to \mathbb{R}^2$



$(1, 1) \overset{\vec{F}}{\leadsto} (-1, 1)$

$(-1, 1) \leadsto (1, 1)$

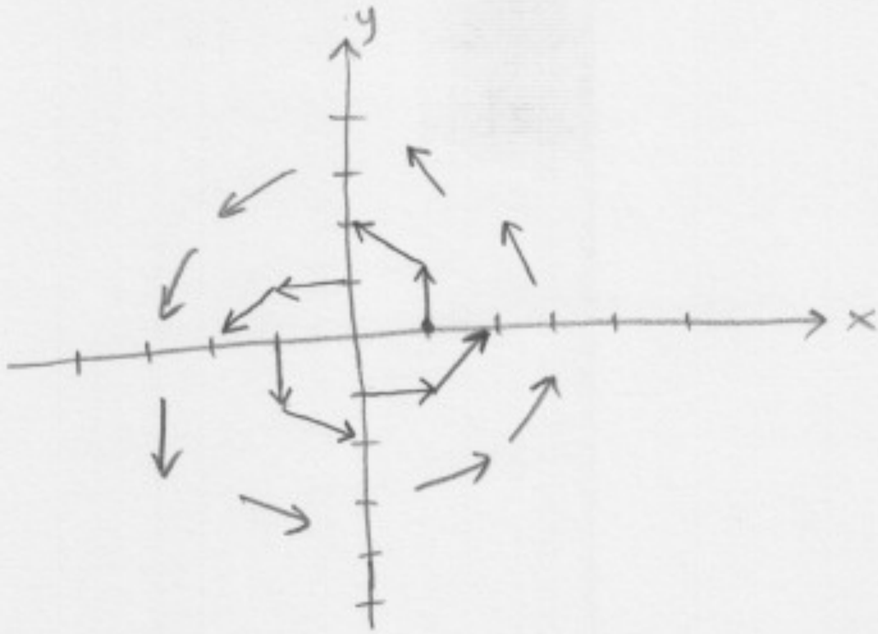
$(1, -1) \to (-1, -1)$

$(-1, -1) \leadsto (1, -1)$

$(2, 1) \to (-2, 1)$

$(2, 2) \to (-2, 2)$

<u>Ex</u>: $\vec{v}(x,y) = (-y, x) : \mathbb{R}^2 \to \mathbb{R}^2$



$\vec{v}$

$(1,0) \to (0,1)$

$(0,1) \to (-1,0)$

$(-1,0) \to (0,-1)$

$(0,-1) \to (1,0)$

$(1,1) \to (-1,1)$

$(-1,-1) \to (1,-1)$

$(-1,1) \to (-1,-1)$

$(1,-1) \to (1,1)$

Gradient vector fields.

<u>Ex</u>: Let $f : \mathbb{R}^3 \to \mathbb{R}$

$$\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}(x,y,z), \frac{\partial f}{\partial y}(x,y,z), \frac{\partial f}{\partial z}(x,y,z)\right) : \mathbb{R}^3 \to \mathbb{R}^3$$

This is an example of vector field, called "<u>Gradient vector field</u>", or "<u>Potential vector field</u>", or "<u>Conservative vector field</u>".

Ex:

$T(x,y,z)$ is the temperature
$\vec{J} = -K\nabla T$ is the heat flux vector field.
$K>0$ is the conductivity

$-\nabla T$ points in the direction of decreasing T.

Ex:

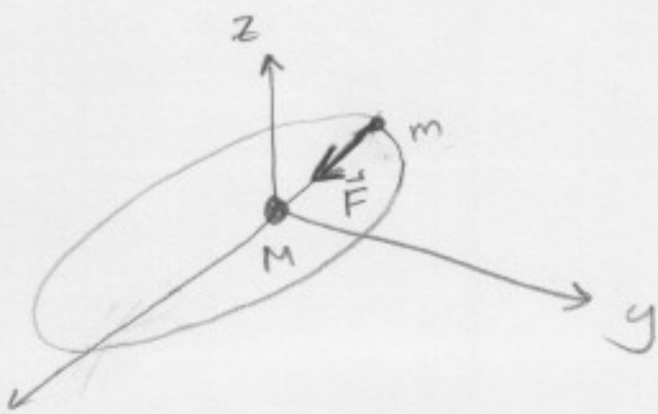
$$\vec{F} = \frac{-mMG}{r^3}\vec{r} = \left(\frac{-mMGx}{r^3}, \frac{-mMGy}{r^3}, \frac{-mMGz}{r^3}\right),$$

$r(x,y,z) = (x^2+y^2+z^2)^{1/2}$

$\vec{F}$ is the gravitational force field. $\vec{F}$ is a gradient vector field since:

$$\vec{F} = -\nabla V,$$

where $V(x,y,z) = \dfrac{-mMG}{\sqrt{x^2+y^2+z^2}}$ is the gravitational potential.



We check:

$$\frac{\partial V}{\partial x} = \frac{\partial}{\partial x}\left(-mMG\left(x^2+y^2+z^2\right)^{-1/2}\right)$$

$$= \frac{mMG}{\cancel{2}}\left(x^2+y^2+z^2\right)^{-3/2}(\cancel{2}x) = \frac{mMGx}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$\frac{\partial V}{\partial y} = \frac{mMGy}{\left(x^2+y^2+z^2\right)^{3/2}} \qquad \frac{\partial V}{\partial z} = \frac{mMGz}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$\therefore \quad \vec{F} = -\nabla V$$

<u>Ex</u>: Consider a particle m moving in a force field $\vec{F}$ that is a potential field (ie., $\vec{F} = -\nabla V$). Show that the energy is conserved.

If the particle follows the path $\vec{r}(t)$, we have, since $F = ma$,

$$m\vec{r}\,''(t) = \vec{F}(\vec{r}(t)) = -\nabla V(\vec{r}(t))$$

$$E(t) = \frac{1}{2}m\|\vec{r}\,'(t)\|^2 + V(\vec{r}(t))$$

$$= \frac{1}{2}m\,\vec{r}\,'(t)\cdot\vec{r}\,'(t) + V(\vec{r}(t))$$

$$\frac{dE}{dt} = \frac{1}{\cancel{2}}m\left(\cancel{2}\,\vec{r}\,''(t)\cdot\vec{r}\,'(t)\right) + \nabla V(\vec{r}(t))\cdot\vec{r}\,'(t)$$

$$= \vec{r}\,'(t)\cdot\left[m\vec{r}\,''(t) + \nabla V(\vec{r}(t))\right]$$

$$= \vec{r}\,'(t)\cdot\left[-\nabla V(\vec{r}(t)) + \nabla V(\vec{r}(t))\right] = 0$$

$\Rightarrow$ Energy is conserved.

<u>Ex</u>: Not every vector field is a gradient vector field.

<u>Ex</u>: Show that $\vec{v}(x,y) = (y, -x)$ is not a gradient vector field; i.e, there is no C^1 function f such that:

$$\vec{v} = \nabla f$$

Suppose that such an f exists. Hence, we have:

$$\Rightarrow \quad \frac{\partial f}{\partial x} = y \qquad \frac{\partial f}{\partial y} = -x$$

$$\Rightarrow \quad \frac{\partial^2 f}{\partial y \partial x} = 1 \quad , \quad \frac{\partial^2 f}{\partial x \partial y} = -1$$

Since $f_{xy} \neq f_{yx}$, we get a contradiction since for f is C^1, the mixed partial should be equal. We conclude that such an f does not exist.

<u>Flow lines</u>:

If $\vec{F}$ is a vector field, a flow line for $\vec{F}$ is a path $\vec{r}(t)$ such that:

$$\vec{r}\,'(t) = \vec{F}(\vec{r}(t)).$$

That is, $\vec{F}$ yields the velocity field of the path $\vec{r}(t)$.

Ex: Verify that $\vec{r}(t) = (\sin t, \cos t, e^t)$ is a flow line of $\vec{v} = (y, -x, z)$.

We check:

$$\vec{r}'(t) = (\cos t, -\sin t, e^t)$$

$$\vec{V}(\vec{r}(t)) = (\cos t, -\sin t, e^t)$$

$$\Rightarrow \vec{r}'(t) = \vec{V}(\vec{r}(t)). \quad \blacksquare$$